# *THE MASSLESS DIRAC EQUATION IN THREE DIMENSIONS: DISPERSIVE ESTIMATES AND ZERO ENERGY OBSTRUCTIONS*

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Abstract. We investigate dispersive estimates for the massless three dimensional Dirac equation with a potential. In particular, we show that the Dirac evolution satisfies a $\langle t\rangle^{-1}$ decay rate as an operator from $L^1$ to $L^\infty$ regardless of the existence of zero energy eigenfunctions. We also show this decay rate may be improved to $\langle t\rangle^{-1-\gamma}$ for any $0\leq\gamma<1/2$ at the cost of spatial weights. This estimate, along with the $L^2$ conservation law allows one to deduce a family of Strichartz estimates in the case of a threshold eigenvalue. We classify the structure of threshold obstructions as being composed of zero energy eigenfunctions. Finally, we show the Dirac evolution is bounded for all time with minimal requirements on the decay of the potential and smoothness of initial data.

## 1. Introduction

We consider the linear Dirac equation with a potential:

$$i\partial_t\psi(x,t)=(D_m+V(x))\psi(x,t),\qquad \psi(x,0)=\psi_0(x).$$

Here, $x\in\mathbb{R}^3$ is the spatial variable and $\psi(x,t)\in\mathbb{C}^4$. The free Dirac operator $D_m$ is defined by

$$D_m=-i\alpha\cdot\nabla+m\beta=-i\sum_{k=1}^{3}\alpha_k\partial_k+m\beta,$$

where $m\geq 0$ is a constant, and the $4\times 4$ Hermitian matrices $\alpha_0:=\beta$ and $\alpha_j$ satisfy

$$\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}\mathbb{1}_{\mathbb{C}^4},\qquad \text{for all } j,k\in\{0,1,2,3\}. \tag{1}$$

We consider the massless case, when $m=0$, which may be used to model the dynamics of massless Fermions. The Dirac equation more generally is a model for relativistic dynamics of quantum particles. We refer to the short introductory article, [25], or the monograph of Thaller, [40], for more detailed introductions to the Dirac equation. For concreteness, in three dimensions we use

$$\beta=\begin{bmatrix} I_{\mathbb{C}^2} & 0\\ 0 & -I_{\mathbb{C}^2}\end{bmatrix},\ \alpha_i=\begin{bmatrix} 0 & \sigma_i\\ \sigma_i & 0\end{bmatrix},$$

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$$\sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The following identity,[1] which follows from (1),

$$(D_m - \lambda \mathbb{1})(D_m + \lambda \mathbb{1}) = (-i\alpha \cdot \nabla + m\beta - \lambda \mathbb{1})(-i\alpha \cdot \nabla + m\beta + \lambda \mathbb{1}) = (-\Delta + m^2 - \lambda^2) \tag{2}$$

allows us to formally define the free Dirac resolvent operator $\mathcal{R}_0(\lambda) = (D_m - \lambda)^{-1}$ in terms of the free resolvent $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ of the Schrödinger operator for $\lambda$ in the resolvent set:

$$\mathcal{R}_0(\lambda) = (D_m + \lambda) R_0(\lambda^2 - m^2). \tag{3}$$

For our purposes, when $m = 0$, we have

$$\mathcal{R}_0(\lambda) = (-i\alpha \cdot \nabla + \lambda) R_0(\lambda^2) = (D_0 + \lambda) R_0(\lambda^2).$$

To state our main theorem, we introduce the following notation. We let $\langle x \rangle = (1 + |x|^2)^{1/2}$, let $a-$ denote $a - \epsilon$ for an arbitrarily small, but fixed $\epsilon > 0$. Similarly, $a+ = a + \epsilon$. We write $A \lesssim B$ if there is an absolute constant $C > 0$ so that $A \leq CB$. For matrix-valued functions if $|V_{ij}(x)| \lesssim \langle x \rangle^{-\delta}$ for every entry, we write $|V(x)| \lesssim \langle x \rangle^{-\delta}$. We define $\chi(\lambda)$ to be a smooth even cut-off to a sufficiently small neighborhood of zero. Similarly, any function space $X$ used in the paper denotes the space of $\mathbb{C}^4$-valued functions with all entries in $X$. That is, by $f \in L^1(\mathbb{R}^3)$ we mean $f(x) = (f_j(x))_{j=1}^4$ with each component an $L^1$ function. We define the polynomially weighted spaces $L^{p,\sigma} = \{f : \langle \cdot \rangle^\sigma f \in L^p\}$. We call the threshold zero energy regular if there are no zero-energy eigenfunctions of the Dirac operator $\mathcal{H} := D_0 + V$. Our main results control the evolution of the Dirac operator.

**Theorem 1.1.** *Assume that $V$ is self-adjoint and $|V(x)| \lesssim \langle x \rangle^{-\delta}$.*

*i) Assume that zero is regular, for fixed $0 \leq \gamma \leq 1$, if $\delta > 3 + 2\gamma$ we have*

$$\|e^{-it\mathcal{H}} \chi(\mathcal{H})\|_{L^{1,\gamma} \to L^{\infty,-\gamma}} \lesssim \langle t \rangle^{-1-\gamma}.$$

*ii) If zero is not regular, then for fixed $0 \leq \gamma < 1/2$, if $\delta > 3 + 4\gamma$ we have*

$$\|e^{-it\mathcal{H}} \chi(\mathcal{H})\|_{L^{1,\gamma} \to L^{\infty,-\gamma}} \lesssim \langle t \rangle^{-1-\gamma}.$$

One main novelty of these results is that the same time decay bounds hold regardless of the regularity of the threshold. This phenomenon, to the best of the authors' knowledge, is found only in one-dimensional cases without additional assumptions on the structure of the threshold eigenspace, see [30, 22] for example. A similar bound holds in the case of the two-dimensional massless case, [17], though one must first subtract off a finite rank operator that decays more slowly for large $t$.

[1] When discussing scalar operators such as $-\Delta + m^2 - \lambda^2$ in the context of the Dirac equation they are to be understood as matrix-valued, that is as $(-\Delta + m^2 - \lambda^2)\mathbb{1}_{\mathbb{C}^4}$.

For completeness, we pair these bounds with an argument that controls the high energy portion of the evolution. We define $P_{ac}$ to be the projection onto the absolutely continuous subspace of $L^2$. We obtain the following family of bounds.

**Theorem 1.2.** *Assume that $V$ is self-adjoint with continuous entries satisfying $|V(x)| \lesssim \langle x\rangle^{-\delta}$. For any fixed $0 \leq \gamma \leq 1$, we have*

$$\|e^{-it\mathcal{H}} P_{ac}(\mathcal{H})\langle \mathcal{H}\rangle^{-3-}\|_{L^{1,\gamma}\to L^{\infty,-\gamma}} \lesssim \langle t\rangle^{-1-\gamma},$$

*provided $\delta > 3+2\gamma$ if zero is regular. If there is an eigenvalue at zero, then for any $0 \leq \gamma < 1/2$, we require $\delta > 3+4\gamma$ .*

We establish the weighted bounds by developing appropriate representations of the spectral measure associated with the perturbed evolution. Finally, as an application of the Lipschitz continuity properties of the spectral measure that we develop, we obtain weaker bounds with minimal decay assumptions on the potential.

**Theorem 1.3.** *Assume that $V$ is self-adjoint with continuous entries satisfying $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 1$. If zero energy is regular, then*

$$\|e^{-it\mathcal{H}} P_{ac}(\mathcal{H})\langle \mathcal{H}\rangle^{-3-}\|_{L^{1}\to L^{\infty}} \lesssim 1.$$

We note that this result is essentially sharp with respect to differentiability of the initial data and required decay at infinity. The free Dirac evolution requires the same amount of differentiability, see Theorem 2.2 below.

The class of potentials considered here are of the form obtained by linearizing about a standing wave solution to a nonlinear Dirac equation, see [9] for example. Our estimates follow by treating the Dirac evolution as an element of the functional calculus. For the potentials we consider $\mathcal{H} = D_0+V$ is self-adjoint, and $\sigma(\mathcal{H}) = \mathbb{R}$. Further, there is a Weyl criterion, $\sigma_c(\mathcal{H}) = \sigma_c(D_0) = \mathbb{R}$, and there is no singularly continuous spectrum or embedded eigenvalues, [28, 7]. The spectral measure may be constructed in terms of the limiting resolvent operators

$$\mathcal{R}_V^{\pm}(\lambda) = \lim_{\epsilon\to 0^+} [D_0 + V - (\lambda \pm i\epsilon)]^{-1}.$$

These operators are well-defined by Agmon's limiting absorption principle as operators on weighted $L^2$ spaces, [1], and their relationship to the Schrödinger resolvents (3). The difference of these limiting operators provide the spectral measure. Specifically, the Stone's formula allows us to express the evolution of the solution operator as

$$e^{-it\mathcal{H}} P_{ac}(\mathcal{H}) f = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda}[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda) f\, d\lambda. \tag{4}$$

Our methods seek to understand the perturbed resolvents $\mathcal{R}_V^{\pm}$ as perturbations of the free resolvent operators

$$\mathcal{R}_0^{\pm}(\lambda) = \lim_{\epsilon \to 0^+} [D_0 - (\lambda \pm i\epsilon)]^{-1}.$$

Using (2), we obtain the identity $\mathcal{R}_0^{\pm}(\lambda) = (D_0 + \lambda) R_0^{\pm}(\lambda)$ where $R_0^{\pm}(\lambda) = (-\Delta - (\lambda \pm i0))^{-1}$ are the limiting resolvent operators for the Schrödinger operator. Due to the well-known explicit formulas for these, one may write explicit formulas for the Dirac resolvent, see (11) below.

Global dispersive bounds that seek to control the $L^\infty$ size of solutions are well-studied in the Schrödinger, wave, and Klein-Gordon contexts, see the recent survey paper [39]. The dispersive estimates for the three dimensional Dirac equation is more studied going back to the work of Boussaïd [6], and D'Ancona and Fanelli, [14] in the massive $m > 0$ in the weighted-$L^2$ setting. The characterization of threshold obstructions as resonances and eigenvalues along with their effect on the dispersive bounds in two and three dimensions has been studied by Erdoğan and the first author in [21], along with Toprak, [23, 24] in the massive case, and with Goldberg in the massless case [17]. More recent work studied the dispersive bounds in the one dimensional case by Erdoğan and the first author in [22]. See also the recent work of Kraisler, Sagiv and Weinstein, [35], which considered non-compact perturbations in one dimension. Much of the work relies on the techniques developed in the study of other dispersive equations, notably the Schrödinger equation [36, 30, 26, 34, 19], which analyze the effect of threshold energy obstructions.

Nonlinear Dirac equations have also garnered interest. See for example, [27, 37, 4, 5, 13, 8], and Boussaïd and Comech's monograph, [9]. There is a longer history in the study of spectral properties of Dirac operators. Limiting absorption principles for the Dirac operators have been studied in [41, 28, 10, 16, 12]. The lack of embedded eigenvalues, singular continuous spectrum and other spectral properties is well established, [3, 28, 2, 12, 7].

There has been recent work on the massless three dimensional Dirac equation with a Coulomb potential of the from $\nu/|x|$. Here one must restrict the value of $\nu$ to an appropriate interval to ensure there is a self-adjoint extension of the Dirac operator. Danesi, [15], and separately Cacciafesta, Séré and Zhang, [11] establish various families of Strichartz estimates for these operators with Coulomb potentials.

Strichartz estimates for potentials of this form are known when zero is regular, [16], in both the massive and massless cases. By interpolating the bound in Theorem 1.2 with the $L^2$ conservation law, one obtains a family of $L^p$ dispersive bounds of the form

$$\|e^{-it\mathcal{H}} \langle \mathcal{H} \rangle^{\frac{3}{2} - \frac{3}{p} -} P_{ac}(\mathcal{H}) f\|_{L^{p'}} \lesssim |t|^{\frac{3}{2} - \frac{3}{p}} \|f\|_{L^p}, \qquad 1 \leq p \leq 2. \tag{5}$$

As in the classic paper of Ginibre and Velo, [29], these may be used to deduce Strichartz estimates.

**Corollary 1.4.** *Assume that $V$ is self-adjoint with continuous entries satisfying $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 3$. If zero energy is not regular, one has*

$$\|e^{-it\mathcal{H}}\langle\mathcal{H}\rangle^{\frac{3}{r}-\frac{3}{2}-}P_{ac}(\mathcal{H})f\|_{L^q_tL^r_x} \lesssim \|f\|_{L^2}, \qquad \frac{2}{q}+\frac{2}{r}=1, \quad q>2,\, 2\le r\le\infty.$$

The paper is organized as follows. First in Section 2 we establish the natural decay properties for the free massless Dirac evolution. In Section 3 we develop expansions of the limiting free resolvent operators in a neighborhood of the threshold, and use them to establish continuity and differentiability properties of the spectral measure near the threshold. In Section 4 we prove the low energy dispersive bounds in Theorem 1.1 when zero is regular. As a further application of the Lipschitz continuity of the spectral measure, we prove families of time-integrable estimates on polynomially weighted space in Subsection 4.1. In Section 5 we show that the same estimates hold even if there is a threshold eigenvalue at the cost of further decay of the potential. In Section 6 we characterize the existence of zero energy eigenvalues and relate them to the spectral measure constructed in Section 3. Finally in Section 7 we control the high energy portion of the evolution.

## 2. Free Dirac dispersive estimates

To understand the dynamics of the perturbed solution operator $e^{-it\mathcal{H}}P_{ac}(\mathcal{H})$, we first study the dynamics of the free solution operator $e^{-itD_0}$ when $V\equiv 0$. In this section we develop the needed oscillatory integral estimates to understand the free evolution, as well as prove several estimates about the dynamics of solutions to the free massless Dirac equation.

Due to the relationship between the massless free Dirac equation and the free wave equation, $D_0^2 f = -\Delta f$, we can expect a natural time decay rate of size $|t|^{-1}$ as one has in the wave equation provided the initial data has more than 2 weak derivatives in $L^1(\mathbb{R}^2)$. In the case of Dirac equation, as in Schrödinger equation, the time decay for large $|t|$ may be improved at the cost of spatial weights.

Much of our low energy analysis will rely on relationships between smoothness of a function and the decay of its Fourier Transform, which we encapsulate in the following.

**Lemma 2.1.** *Let $\mathcal{E}(\lambda)$ be a a function supported on $(-1,1)$ with $\mathcal{E}(\lambda)$ bounded and $\mathcal{E}'(\lambda)\in L^1$. Then, we have the bound*

$$\left|\int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}(\lambda)\,d\lambda\right| \lesssim \frac{1}{|t|}\int_{\mathbb{R}}\left|\mathcal{E}'(\lambda)-\mathcal{E}'\Big(\lambda-\frac{\pi}{t}\Big)\right|d\lambda.$$

*Proof.* Since $\mathcal{E}(\lambda)$ is bounded and $\mathcal{E}'\in L^1$ we may integrate by parts once, the support of $\mathcal{E}$ ensures there are no boundary terms,

$$\int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}(\lambda)\,d\lambda = \frac{1}{it}\int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}'(\lambda)\,d\lambda.$$

We note that if $|t| < 1$ then $\mathcal{E}'(\lambda - \pi/t) = 0$ on the support of $\mathcal{E}(\lambda)$. Applying the triangle inequality to the equality above proves the claim. For $|t| > 1$ large, we use the support of $\mathcal{E}$, a change of variables $\lambda \mapsto \lambda - \pi/t$, and then the fact that $e^{i\pi} = -1$ to write

$$\int_{\mathbb{R}} e^{-it\lambda} \mathcal{E}'(\lambda)\, d\lambda = \int_{\mathbb{R}} e^{-it(\lambda - \pi/t)} \mathcal{E}'(\lambda - \pi/t)\, d\lambda = -\int_{\mathbb{R}} e^{-it\lambda} \mathcal{E}'\Big(\lambda - \frac{\pi}{t}\Big)\, d\lambda.$$

Then using the triangle inequality, we have

$$\begin{aligned}\Big| \int_{\mathbb{R}} e^{-it\lambda} \mathcal{E}(\lambda)\, d\lambda \Big| &= \Big| \frac{1}{it} \int_{\mathbb{R}} e^{-it\lambda} \mathcal{E}'(\lambda)\, d\lambda \Big| = \Big| \frac{1}{2it} \int_{\mathbb{R}} e^{-it\lambda} \Big[ \mathcal{E}'(\lambda) - \mathcal{E}'\Big(\lambda - \frac{\pi}{t}\Big)\Big]\, d\lambda \Big| \\ &\lesssim \frac{1}{|t|} \int_{\mathbb{R}} \Big| \mathcal{E}'(\lambda) - \mathcal{E}'\Big(\lambda - \frac{\pi}{t}\Big) \Big|\, d\lambda\end{aligned}$$

as desired. □

Using this oscillatory integral bound, we can prove dispersive bounds for the free Dirac evolution.

**Theorem 2.2.** *We have the estimate*

$$\big\| e^{-itD_0} \langle D_0 \rangle^{-2-} \big\|_{L^1 \to L^\infty} \lesssim |t|^{-1}.$$

*Furthermore, for $|t| > 1$ and any $0 \le \gamma \le 1$, we have*

$$\big\| \langle x \rangle^{-\gamma} e^{-itD_0} \langle D_0 \rangle^{-2-} \langle y \rangle^{-\gamma} \big\|_{L^1 \to L^\infty} \lesssim |t|^{-1-\gamma}.$$

*Proof.* First note that the Stone's formula, (4), for the free evolution is

$$e^{-itD_0} = \int_{\mathbb{R}} e^{-it\lambda} [\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda)(x,y)\, d\lambda. \tag{6}$$

Write $\mu(\lambda, x, y) := [\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda)(x,y)$. Utilizing (3) and the well-known expansion for the limiting resolvents of the free Schrödinger in three dimensions, see (11) below, we have

$$\mu(\lambda, x, y) = (-i\alpha \cdot \nabla + \lambda) \bigg( \frac{i \sin(\lambda |x-y|)}{2\pi |x-y|} \bigg).$$

Similar to the estimates we establish in Lemma 3.1 below, we have the following bounds:

$$|\mu(\lambda, x, y)| \lesssim \min\bigg( |\lambda|^2, \frac{|\lambda|}{|x-y|} \bigg), \quad |\partial_\lambda \mu(\lambda, x, y)| \lesssim |\lambda|, \quad |\partial_\lambda^2 \mu(\lambda, x, y)| \lesssim 1 + |\lambda|\, |x-y|. \tag{7}$$

For the first bound (with $r = |x-y|$ and $\hat{e} = \nabla_x |x-y| = (x-y)/|x-y|$ the unit vector in the $x-y$ direction) we have

$$\mu(\lambda, x, y) = \alpha \cdot \hat{e}\, \lambda^2 \bigg( \frac{\lambda r \cos(\lambda r) - \sin(\lambda r)}{2\pi (\lambda r)^2} \bigg) + \frac{i\lambda \sin(\lambda r)}{2\pi r}.$$

This is bounded by $|\lambda|/r$, which we obtain by taking $|\cos(\lambda r)| \lesssim 1$ and $|\sin(\lambda r)| \lesssim |\lambda r|$ in the first term and $|\sin(\lambda r)| \lesssim 1$ in the second term. When $|\lambda| r \ge 1$, this is bounded by $|\lambda|^2$ as we may freely multiply by $|\lambda| r$ on the upper bound. When $|\lambda| r < 1$, we utilize the cancellation of the numerator of the first term up to order $(\lambda r)^3$ to see this is bounded by $|\lambda|^3 r \le |\lambda|^2$ as desired.

The contribution of the second term is more easily seen to be of size $|\lambda|^2$ using $|\sin(\lambda r)| \lesssim |\lambda| r$. These bounds hold for any $\lambda$.

For the derivative, we note that

$$2\pi\partial_\lambda \mu(\lambda, x, y) = -i\alpha \cdot \nabla\big(i\cos(\lambda r)\big) + i\lambda\cos(\lambda r) + \frac{i\sin(\lambda r)}{r}$$
$$= (\alpha \cdot \hat{e})\lambda \sin(\lambda r) + i\lambda\cos(\lambda r) + \frac{i\sin(\lambda r)}{r}$$

This is easily seen to be bounded by $|\lambda|$. Finally,

$$|2\pi\partial_\lambda^2 \mu(\lambda, x, y)| = |(\alpha \cdot \hat{e})(\sin(\lambda r) + \lambda r\cos(\lambda r)) + 2i\cos(\lambda r) - i\lambda r\sin(\lambda r)| \lesssim 1 + |\lambda|\, r.$$

For the low energy, we define $\mu_0(\lambda, x, y) = \chi(\lambda)\mu(\lambda, x, y)$, so $\mu_0$ and its derivatives satisfy the corresponding bounds above for derivatives of $\mu$ in (7). If any derivative acts on the cut-off $\chi(\lambda)$, we note that $\chi'(\lambda)$ is supported on the annulus $|\lambda| \approx 1$, so that $|\chi^{(k)}(\lambda)| \lesssim 1$ or $|\lambda|^{-k}$ as needed.

Using (7) and the support of $\chi$, which is contained in $[-1, 1]$, we have

$$\left| \int_{\mathbb{R}} e^{-it\lambda}\mu_0(\lambda, x, y)\, d\lambda \right| \lesssim \int_{-1}^{1} \lambda^2 \; d\lambda,$$

so this integral is bounded uniformly in $x, y$. To obtain time decay, we integrate by parts once. Again we use (7), this time to ensure there are no boundary terms

$$\left| \int_{\mathbb{R}} e^{-it\lambda}\mu_0(\lambda, x, y)\, d\lambda \right| = \left| \frac{1}{it} \int_{\mathbb{R}} e^{-it\lambda}\partial_\lambda\mu_0(\lambda, x, y)\, d\lambda \right| \lesssim \frac{1}{|t|} \int_{-1}^{1} |\lambda|\, d\lambda \lesssim \frac{1}{|t|}.$$

Here, (7) was used along with the support of $\chi$. For the low energy contribution, we note that the bounds of 1 and $|t|^{-1}$ show that it is bounded by $\langle t\rangle^{-1}$. That is, the low energy contribution is bounded for all times.

For the weighted bound, rather than integrate by parts again, we employ an argument based on Lipschitz continuity, which will be helpful for the perturbed case. Take $|\lambda_1| \leq |\lambda_2| \leq 1$. We claim the following bound holds on the support of $\chi$:

$$|\mu_0(\lambda_2, x, y) - \mu_0(\lambda_1, x, y)| \lesssim |\lambda_2|^{2-\gamma}|\lambda_2 - \lambda_1|^\gamma, \qquad 0 \leq \gamma \leq 1. \tag{8}$$

By the triangle inequality and (7) we have

$$|\mu_0(\lambda_2, x, y) - \mu_0(\lambda_1, x, y)| \lesssim |\lambda_2|^2.$$

On the other hand if we apply the mean value theorem and (7), writing $I = [\min(\lambda_1, \lambda_2), \max(\lambda_1, \lambda_2)]$, we see

$$|\mu_0(\lambda_2, x, y) - \mu_0(\lambda_1, x, y)| = \left| \int_{\lambda_1}^{\lambda_2} \partial_\lambda \mu_0(s, x, y)\, ds \right|$$
$$\lesssim |\lambda_2 - \lambda_1| \sup_{s\in I} |\partial_\lambda\mu_0(s, x, y)| \lesssim |\lambda_2|\, |\lambda_2 - \lambda_1|.$$

Interpolating between these two bounds gives the claim in (8).

On the support of $\chi$, a similar argument shows that

$$|\partial_\lambda \mu_0(\lambda_2, x, y) - \partial_\lambda \mu_0(\lambda_1, x, y)| \lesssim |\lambda_2|^{1-\gamma}|\lambda_2 - \lambda_1|^\gamma (1 + |\lambda_2|\,|x-y|)^\gamma, \qquad 0 \le \gamma \le 1. \tag{9}$$

Now, for the weighted bound if $|t| > 1$ we apply Lemma 2.1 (using (9) with $\mathcal{E} = \mu_0$, $\lambda_j$ one of $\lambda$ and $\lambda - \pi/t$) to see

$$\left| \int_{\mathbb{R}} e^{-it\lambda} \mu_0(\lambda, x, y)\, d\lambda \right| \lesssim \frac{1}{|t|} \int_{-1}^{1} |t|^{-\gamma}|x-y|^\gamma \, d\lambda \lesssim |t|^{-1-\gamma}|x-y|^\gamma.$$

Note that $|x-y| \lesssim \langle x\rangle\langle y\rangle$ suffices to establish the claim for low energy, when $\lambda$ is in a neighborhood of zero. For large $|\lambda|$, we define the complementary cut-off $\widetilde{\chi}(\lambda) = 1 - \chi(\lambda)$. We need to bound

$$\int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda)\langle\lambda\rangle^{-2-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)\, d\lambda.$$

By (7) this integral is not absolutely convergent uniformly in $x, y$, but these bounds suffice to ensure there are no boundary terms when integrating by parts.

$$\left| \int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda)\langle\lambda\rangle^{-2-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)\, d\lambda \right| \lesssim \frac{1}{|t|} \int_{\mathbb{R}} \left| \partial_\lambda [\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-2-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)] \right| d\lambda.$$

We note that, on the support of $\widetilde{\chi}$, differentiation of the first two terms is comparable to division by $\lambda$. Hence, we may bound the above integral by

$$\frac{1}{|t|} \int_{\mathbb{R}} \langle\lambda\rangle^{-1-}\, d\lambda \lesssim \frac{1}{|t|}.$$

Integrating by parts twice leads to the bound

$$\begin{aligned} \left| \int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda)\langle\lambda\rangle^{-2-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)\, d\lambda \right| &\lesssim \frac{1}{|t|^2} \int_{\mathbb{R}} \left| \partial_\lambda^2 [\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-2-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)] \right| d\lambda \\ &\lesssim \frac{1}{|t|^2} \int_{\mathbb{R}} (\langle\lambda\rangle^{-2-} + \langle\lambda\rangle^{-1-}|x-y|)\, d\lambda \lesssim \frac{\langle x-y\rangle}{|t|^2}. \end{aligned}$$

Noting that $\langle x-y\rangle \lesssim \langle x\rangle\langle y\rangle$ and interpolating between the two bounds establishes the claim. □

We utilized an argument based on Lipschitz continuity of the resolvents here rather than integrating by parts twice since such an argument is beneficial in the analysis of the perturbed operator. Utilizing Lipschitz continuity of the perturbed resolvent in a neighborhood of $\lambda = 0$ will allow us to obtain faster time decay with minimal further assumptions on the decay of the potential.

The small time blow-up is a high energy phenomenon. We may also utilize the techniques in the proof above to obtain weaker dispersive bounds that require more smoothness on the initial data to control the high energy.

**Corollary 2.3.** *We have the bound*

$$\|e^{-itD_0}\langle D_0\rangle^{-3-}\|_{L^1\to L^\infty} \lesssim \langle t\rangle^{-1}.$$

*Proof.* Note that the extra power of $\langle D_0\rangle^{-1}$ is needed to ensure that the integral

$$\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda,x,y)\,d\lambda\right|,$$

converges absolutely, using $|\mu(\lambda)(x,y)| \lesssim |\lambda|^2$. This, combined with Theorem 2.2 suffices to prove the claim. □

## 3. Low Energy Resolvent Expansions and Estimates

In this section we show that, under mild decay assumptions on the potential $V$, the low energy evolution of the perturbed solution operator obeys the same bounds as the free evolution. We do this by developing expansions for the spectral measure $[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)$ in a neighborhood of the threshold to use in the Stone's formula, (4). In this low energy regime, we treat $\mathcal{R}_V^\pm$ as a perturbation of the free resolvent $\mathcal{R}_0^\pm$, for which we obtain explicit asymptotic formulas.

To obtain expansions for the perturbed resolvent operators $\mathcal{R}_V^\pm(\lambda)$, we recall the symmetric resolvent identity. As in previous analyses of the Dirac Equation, [21, 23, 24, 17], using that $V\colon \mathbb{R}^3 \to \mathbb{C}^{4\times 4}$ is self-adjoint, the spectral theorem for self-adjoint matrices allows us to write

$$\begin{aligned} V &= B^*\Lambda B = B^*|\Lambda|^{\frac12}U|\Lambda|^{\frac12}B =: v^*Uv, \quad \text{where} \\ \Lambda &= \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4), \text{ with } \lambda_j\in\mathbb{R}, \\ |\Lambda|^{\frac12} &= \operatorname{diag}(|\lambda_1|^{\frac12},|\lambda_2|^{\frac12},|\lambda_3|^{\frac12},|\lambda_4|^{\frac12}), \\ U &= \operatorname{diag}(\operatorname{sign}(\lambda_1),\operatorname{sign}(\lambda_2),\operatorname{sign}(\lambda_3),\operatorname{sign}(\lambda_4)). \end{aligned}$$

We note that if the entries of $V(x)$ are all bounded by $\langle x\rangle^{-\delta}$, then the entries of $v(x)$ and $v^*(x)$ are bounded by $\langle x\rangle^{-\delta/2}$. Defining $M^\pm(\lambda) = U + v\mathcal{R}_0^\pm(\lambda)v^*$, the symmetric resolvent identity yields

$$\mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)v^*(M^\pm)^{-1}(\lambda)v\mathcal{R}_0^\pm(\lambda). \tag{10}$$

We consider $M^\pm(\lambda)$ as a perturbation of $M^\pm(0) = U + v\mathcal{R}_0^\pm(0)v^* = U + v\mathcal{G}_0v^*$. We denote this operator by $T_0 := M^\pm(0)$.

We recall the expansion of the free resolvent for the Schrödinger operator in $\mathbb{R}^3$, see [26] for example, which has

$$R_0^\pm(\lambda^2)(x,y) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|} = \sum_{j=0}^{J}(\pm i\lambda)^jG_j + o(\lambda^J),$$

where we define the scalar-valued operators $G_j = \frac{1}{4\pi j!}|x-y|^{j-1}$. To write expansions for the Dirac resolvent, we use (3) and note that

$$\mathcal{R}_0^\pm(\lambda)(x,y) = (-i\alpha\cdot\nabla+\lambda)\left(\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}\right) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}\left(\alpha\cdot\hat{e}\left(\pm\lambda + \frac{i}{|x-y|}\right)+\lambda\right). \tag{11}$$

Recall $\hat{e} := (x-y)/|x-y|$ is the unit vector in the $x-y$ direction. This directly gives the bounds

$$|\mathcal{R}_0^{\pm}(\lambda,x,y)| \lesssim \frac{1}{|x-y|^2} + \frac{|\lambda|}{|x-y|} \quad \text{and} \quad |\partial_\lambda \mathcal{R}_0^{\pm}(\lambda,x,y)| \lesssim \frac{1}{|x-y|} + |\lambda|. \tag{12}$$

Now, we define operators $\mathcal{G}_j$ in terms of their integral kernels:

$$\mathcal{G}_0(x,y) = \frac{i\alpha\cdot\hat{e}}{4\pi|x-y|^2}, \qquad \mathcal{G}_1(x,y) = \frac{1}{4\pi|x-y|}, \qquad \mathcal{G}_2^{\pm}(x,y) = \pm\frac{1}{4\pi} + \frac{\alpha\cdot\hat{e}}{8\pi}. \tag{13}$$

Recall that these kernels are matrix-valued operators. $\mathcal{G}_1$ and $\mathcal{G}_2^{\pm}$ are $4\times 4$ matrices, since $\alpha\cdot\hat{e} = \sum_{j=1}^{3}\alpha_j e_j$ is a $4\times 4$ matrix. Further, we note that $\mathcal{G}_1(x,y) = (-\Delta)^{-1}(x,y)\mathbb{1}_{\mathbb{C}^4}$.

**Lemma 3.1.** *We have the following expansions for the Dirac free resolvents:*

$$\begin{aligned}\mathcal{R}_0^{\pm} &= \mathcal{G}_0 + \lambda\mathcal{G}_1 + \mathcal{E}_1^{\pm}(\lambda,|x-y|)\\ &= \mathcal{G}_0 + \lambda\mathcal{G}_1 + i\lambda^2\mathcal{G}_2^{\pm} + \mathcal{E}_2^{\pm}(\lambda,|x-y|),\end{aligned}$$

*where*

$$|\mathcal{E}_1^{\pm}(\lambda,|x-y|)| \lesssim \frac{|\lambda|}{|x-y|}, \quad |\partial_\lambda\mathcal{E}_1^{\pm}(\lambda,|x-y|)| \lesssim |\lambda|, \quad |\partial_\lambda^2\mathcal{E}_1^{\pm}(\lambda,|x-y|)| \lesssim 1 + |\lambda|\,|x-y|.$$

*Further, for any choice of* $0\le\ell\le 1$*, we have*

$$|\mathcal{E}_2^{\pm}(\lambda,|x-y|)| \lesssim |\lambda|^2(|\lambda|\,|x-y|)^{\ell}, \qquad |\partial_\lambda\mathcal{E}_2^{\pm}(\lambda,|x-y|)| \lesssim |\lambda|(|\lambda|\,|x-y|)^{\ell},$$
$$|\partial_\lambda^2\mathcal{E}_2^{\pm}(\lambda,|x-y|)| \lesssim 1 + |\lambda|\,|x-y|.$$

*Proof.* Using the Taylor expansion of $\mathcal{R}_0^{\pm}(\lambda)(x,y)$ in $|\lambda||x-y|$, we see

$$\begin{aligned}\mathcal{R}_0^{\pm}(\lambda)(x,y) &= (-i\alpha\cdot\nabla + \lambda)\left(\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}\right)\\ &= \frac{i\alpha\cdot\hat{e}}{4\pi|x-y|^2} + \frac{\lambda}{4\pi|x-y|} \pm \frac{i\lambda^2}{8\pi} + \frac{i\lambda^2\alpha\cdot\hat{e}}{8\pi} + \mathcal{E}_2^{\pm}(\lambda,|x-y|)\\ &= \mathcal{G}_0(x,y) + \lambda\mathcal{G}_1(x,y) + i\lambda^2\mathcal{G}_2^{\pm}(x,y) + \mathcal{E}_2^{\pm}(\lambda,|x-y|).\end{aligned}$$

where $\mathcal{E}_2^{\pm}$ may be differentiated freely, with differentiation comparable to division by $\lambda$.

Define $\mathcal{E}_1^{\pm}(\lambda,|x-y|) = \mathcal{R}_0^{\pm}(\lambda)(x,y) - \mathcal{G}_0(x,y) - \lambda\mathcal{G}_1(x,y)$. Denote $r := |x-y|$, the Taylor expansion about $\lambda r = 0$ gives the following bounds when $|\lambda|r < 1$:

$$|\mathcal{E}_1^{\pm}(\lambda,r)| \lesssim |\lambda|^2, \qquad |\partial_\lambda\mathcal{E}_1^{\pm}(\lambda,r)| \lesssim |\lambda|, \qquad |\partial_\lambda^2\mathcal{E}_1^{\pm}(\lambda,r)| \lesssim 1. \tag{14}$$

For large $|\lambda|r$, using (11) when $|\lambda|r \gtrsim 1$, we see that

$$\mathcal{R}_0^{\pm}(\lambda)(x,y) - \mathcal{G}_0 = \frac{e^{\pm i\lambda r}}{4\pi r}(\pm\lambda\alpha\cdot\hat{e} + \lambda) + \frac{\alpha\cdot\hat{e}}{4\pi}\left(\frac{e^{\pm i\lambda r} - 1}{r^2}\right).$$

Using that $|e^{\pm i\lambda r}-1|\lesssim|\lambda|r$, we see that this piece may be bounded in modulus by $|\lambda|r^{-1}$, as can the contribution of $\lambda\mathcal{G}_1$. For derivatives, we have that differentiation is bounded by multiplication by $r$, so that

$$|\partial_\lambda^k\mathcal{R}_0^\pm(\lambda)(x,y)|\lesssim\left(\frac{|\lambda|}{r}+\frac{1}{r^2}\right)r^k\lesssim|\lambda|r^{k-1}+r^{k-2}. \tag{15}$$

Now explicitly writing $\mathcal{E}_1^\pm(\lambda,r)$, we have $|\mathcal{E}_1^\pm(\lambda,r)|=|\mathcal{R}_0^\pm(\lambda,x,y)-\mathcal{G}_0(x,y)-\lambda\mathcal{G}_1(x,y)|\lesssim|\lambda|r^{-1}$. Since $\mathcal{G}_0$ is $\lambda$ independent, derivatives are controlled by (15). So that, when $|\lambda|r\gtrsim 1$ we have

$$|\partial_\lambda^k\mathcal{E}_1^\pm(\lambda,r)|\lesssim|\lambda|r^{k-1}+r^{k-2},\qquad k=0,1,2.$$

The bounds on $|\lambda|r\gtrsim 1$ may be freely multiplied by positive powers of $|\lambda|r$, while the bounds in (14) my be divided by powers of $|\lambda|r$ to obtain the bounds in the claim.

The argument for $\mathcal{E}_2^\pm(\lambda,x,y)$ proceeds similarly noting that $\mathcal{E}_2^\pm(\lambda,x,y):=\mathcal{E}_1^\pm(\lambda,x,y)-i\lambda^2\mathcal{G}_2^\pm(x,y)$. When $|\lambda|r<1$, the Taylor expansion gives an error of size $|\lambda|^3r$, which may be differentiated freely. When $|\lambda|r>1$, we note that

$$|\partial_\lambda^k(i\lambda^2\mathcal{G}_2^\pm(x,y))|\lesssim\lambda^{2-k},\qquad k=0,1,2.$$

Combining this with the estimates for $\mathcal{E}_1^\pm(\lambda,r)$ suffices to prove the claim. □

These bounds may be used to obtain Lipschitz bounds on the error term.

**Lemma 3.2.** *Let $|\lambda_1|\le|\lambda_2|\le 1$, then we have the following Lipschitz bounds for the error terms. For any choice of $0\le\gamma\le 1$ we have:*

$$|\mathcal{E}_1^\pm(\lambda_2,|x-y|)-\mathcal{E}_1^\pm(\lambda_1,|x-y|)|\lesssim|\lambda_2-\lambda_1|^\gamma\left(\frac{|\lambda_2|}{|x-y|^{1-\gamma}}\right),$$
$$|\partial_\lambda\mathcal{E}_1^\pm(\lambda_2,|x-y|)-\partial_\lambda\mathcal{E}_1^\pm(\lambda_1,|x-y|)|\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_2|^{1-\gamma}(1+|x-y|^\gamma).$$

*Similarly, for any $0\le\ell\le 1$*

$$|\mathcal{E}_2^\pm(\lambda_2,|x-y|)-\mathcal{E}_2^\pm(\lambda_1,|x-y|)|\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_2|^{1+\gamma+\ell}|x-y|^\ell,$$
$$|\partial_\lambda\mathcal{E}_2^\pm(\lambda_2,|x-y|)-\partial_\lambda\mathcal{E}_2^\pm(\lambda_1,|x-y|)|\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_2|^{(1+\ell)(1-\gamma)}(1+|x-y|^{\gamma+\ell(1-\gamma)}).$$

*Proof.* We prove the claim for $\mathcal{E}_1^\pm$, the proof for $\mathcal{E}_2^\pm$ is similar, but simpler. The proof is independent of the $\pm$, so we drop the superscript. We begin by bounding $|\mathcal{E}_1(\lambda_1,|x-y|)-\mathcal{E}_1(\lambda_2,|x-y|)|$. We use the triangle inequality and Lemma 3.1 to obtain

$$|\mathcal{E}_1(\lambda_1,|x-y|)-\mathcal{E}_1(\lambda_2,|x-y|)|\lesssim\frac{|\lambda_2|}{|x-y|},$$

Using the mean value theorem, we obtain the alternative bound

$$|\mathcal{E}_1(\lambda_1,|x-y|)-\mathcal{E}_1(\lambda_2,|x-y|)|\lesssim|\lambda_2-\lambda_1|\sup_{\lambda\in I}|\partial_\lambda\mathcal{E}_1(\lambda,|x-y|)|\lesssim|\lambda_2-\lambda_1||\lambda_2|.$$

Here $I$ is the interval $[\min(\lambda_1,\lambda_2),\max(\lambda_1,\lambda_2)]$. Via interpolation, we obtain

$$|\mathcal{E}_1(\lambda_1,|x-y|)-\mathcal{E}_1(\lambda_2,|x-y|)|\lesssim|\lambda_2-\lambda_1|^{\gamma}\frac{|\lambda_2|}{|x-y|^{1-\gamma}}.$$

We use the same approach for $\partial_\lambda\mathcal{E}_1(\lambda,|x-y|)$. These give us the two bounds

$$\begin{aligned}|\partial_\lambda\mathcal{E}_1(\lambda_1,|x-y|)-\partial_\lambda\mathcal{E}_1(\lambda_2,|x-y|)|&\lesssim|\lambda_2|\\|\partial_\lambda\mathcal{E}_1(\lambda_1,|x-y|)-\partial_\lambda\mathcal{E}_1(\lambda_2,|x-y|)|&\lesssim|\lambda_2-\lambda_1|\sup_{\lambda\in I}\left|\partial_\lambda^2\mathcal{E}_1(\lambda,|x-y|)\right|\\&\lesssim|\lambda_2-\lambda_1|(1+|\lambda_2||x-y|).\end{aligned}$$

We can then interpolate between these bounds to obtain

$$\begin{aligned}|\partial_\lambda\mathcal{E}_1(\lambda_1,x,y)-\partial\lambda\mathcal{E}_1(\lambda_2,x,y)|&\lesssim|\lambda_2-\lambda_1|^{\gamma}(1+|\lambda_2||x-y|)^{\gamma}|\lambda_2|^{1-\gamma}\\&\lesssim|\lambda_2-\lambda_1|^{\gamma}(|\lambda_2|^{1-\gamma}+|\lambda_2||x-y|^{\gamma}).\end{aligned}$$

□

Combining Lemma 3.2 with the first expansion in Lemma 3.1, since the first term is independent of $\lambda$ we have

$$\mathcal{R}_0^{\pm}(\lambda_2)(x,y)-\mathcal{R}_0^{\pm}(\lambda_1)(x,y)=(\lambda_2-\lambda_1)\mathcal{G}_1+\mathcal{E}_1^{\pm}(\lambda_2,|x-y|)-\mathcal{E}_1^{\pm}(\lambda_1,|x-y|).$$

We note that the $\mathcal{G}_1$ terms cancel in the difference of the derivatives. From this we can see

**Corollary 3.3.** *Lipschitz bounds for $\mathcal{E}_1^{\pm}(\lambda,|x-y|)$ also hold for the Dirac resolvent. Namely, for $|\lambda_1|\leq|\lambda_2|\leq 1$ we have*

$$\begin{aligned}|\mathcal{R}_0^{\pm}(\lambda_2)(x,y)-\mathcal{R}_0^{\pm}(\lambda_1)(x,y)|&\lesssim\frac{|\lambda_2-\lambda_1|}{|x-y|}+|\lambda_2-\lambda_1|^{\gamma}\left(\frac{|\lambda_2|}{|x-y|^{1-\gamma}}\right).\\|\partial_\lambda\mathcal{R}_0^{\pm}(\lambda_2)(x,y)-\partial_\lambda\mathcal{R}_0^{\pm}(\lambda_1)(x,y)|&\lesssim|\lambda_2-\lambda_1|^{\gamma}|\lambda_2|^{1-\gamma}(1+|x-y|^{\gamma}).\end{aligned}$$

To control the evolution of the perturbed solution operator, we must distinguish between the cases when zero energy is regular and exceptional. To that end, we make the following definition, which is similar to the definitions for the massive cases [21, 23, 24, 22] and massless case [17] respectively. These had roots in earlier works on the Schrödinger operators such as [26, 19, 20].

**Definition 3.4.** *We make the following definitions that characterize zero energy obstructions.*

*i) We define zero energy to be regular if $T_0=M^{\pm}(0)$ is invertible on $L^2(\mathbb{R}^3)$.*

*ii) We say that zero is not regular if $T_0$ is not invertible on $L^2(\mathbb{R}^3)$. In this case we define $S_1$ is the Riesz projection onto the kernel of $T_0$, so that $T_0+S_1$ is invertible on $L^2(\mathbb{R}^3)$. We show, in Section 6, the connection of zero energy not being regular to the existence of zero energy eigenvalues of $\mathcal{H}=D_0+V$.*

*iii) Noting that $v\mathcal{G}_0v^*$ is compact and self-adjoint, it follows that $T_0 = U + v\mathcal{G}_0v^*$ is a compact perturbation of $U$. Since the spectrum of $U$ is in $\{\pm 1\}$, zero is an isolated point of the spectrum of $T_0$ and the kernel is a finite-dimensional subspace of $L^2(\mathbb{R}^3)$. It then follows that $S_1$ is a finite rank projection.*

We employ the following terminology from [38, 19, 20]:

**Definition 3.5.** *We say an operator $T\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with kernel $T(\,\cdot\,,\,\cdot\,)$ is absolutely bounded if the operator with kernel $|T(\,\cdot\,,\,\cdot\,)|$ is bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$.*

Recall the Hilbert-Schmidt norm of an operator $T$ with integral kernel $T(x,y)$ is defined by

$$\|T\|_{HS}^2 = \int_{\mathbb{R}^6} |T(x,y)|^2\, dx\, dy.$$

Recall that Hilbert-Schmidt and finite-rank operators are absolutely bounded.

We now proceed to building expansions for the operators $M^{\pm}(\lambda)$ when zero energy is regular. The following estimate is used frequently.

**Lemma 3.6.** *Fix $x, y \in \mathbb{R}^n$, with $0 \le k, \ell < n$, $\delta > 0$, $k+\ell+\delta \ge n$, $k+\ell \neq n$:*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\delta-}}{|z-x|^k|y-z|^\ell} dz \lesssim \begin{cases} |x-y|^{-\max\{0,k+\ell-n\}} & \text{if } |x-y|<1; \\ |x-y|^{-\min\{k,\ell,k+\ell+\delta-n\}} & \text{if } |x-y|\ge 1. \end{cases}$$

*Consequently, we have that*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\delta-}}{|z-x|^k|y-z|^\ell} dz \lesssim \frac{1}{|x-y|^p} \tag{16}$$

*where we may take any $p \in [\max\{0, k+\ell-n\}, \min\{k,\ell,k+\ell+\delta-n\}]$ as desired.*

Note that for $a, b, \varepsilon > 0$ and $\ell > k$, we have $a^{-k}b^{-\ell} \lesssim a^{-k}(b^{-\ell-\varepsilon} + b^{-\ell+\varepsilon})$, which allows for the application of Lemma 3.6 when $k+\ell = n = 3$.

*Proof.* The first claim is Lemma 6.3 in [18]. The second claim follows by noting that $\min\{k,\ell,k+\ell+\delta-n\} \ge \max\{0,k+\ell-n\}$ noting that if $|x-y|<1$ selecting $p \ge \max\{0,k+\ell-n\}$ increases the upper bound, while if $|x-y|>1$ selecting $p \le \min\{k,\ell,k+\ell+\delta-n\}$ also increases the upper bound. □

To handle the case when either $k$ or $\ell = 0$, we recall Lemma 3.8 in [31], which we state in less generality:

**Lemma 3.7.** *Suppose that $0 \le \delta, k < 3$ with $k+\delta > 3$, then*

$$\int_{\mathbb{R}^3} \frac{\langle x\rangle^{-\delta}}{|x-y|^k}\, dx \lesssim \langle y\rangle^{3-k-\delta}.$$

**Lemma 3.8.** *Assume that $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 0$. We have the expansion*

$$M^{\pm}(\lambda) = T_0 + \lambda v\mathcal{G}_1 v^* + M_0^{\pm}(\lambda),$$

*where, for $\lambda$ in a neighborhood of zero, we have $\|M_0(\lambda)\|_{HS} < \infty$ provided $\delta > 1$, $\|\partial_\lambda M_0(\lambda)\|_{HS} \lesssim |\lambda| < \infty$ provided $\delta > 3$, and $\|\partial_\lambda^2 M_0(\lambda)\|_{HS} < \infty$ provided $\delta > 5$. Furthermore, for any choice of $0 \le \gamma \le 1$ and for $|\lambda_1| \le |\lambda_2| \le 1$, we have*

$$\|M_0^{\pm}(\lambda_2) - M_0^{\pm}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma}, \qquad \text{provided } \delta > 1 + 2\gamma.$$

*Furthermore, if $\delta > 3 + 2\gamma$ we have*

$$\|\partial_\lambda M_0^{\pm}(\lambda_2) - \partial_\lambda M_0^{\pm}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma}.$$

*The Lipschitz bounds also hold for $M^{\pm}(\lambda)$ in place of $M_0^{\pm}(\lambda)$.*

*Proof.* Recall the definition of $M^{\pm}$ as well as $\mathcal{R}_0^{\pm}$

$$M^{\pm}(\lambda) = U + v\mathcal{R}_0^{\pm}(\lambda)v^* = U + v\mathcal{G}_0 v^* + \lambda v\mathcal{G}_1 v^* + v\mathcal{E}_1^{\pm}(\lambda)v^* = U + v\mathcal{G}_0 v^* + M_0^{\pm}(\lambda).$$

To compute the Hilbert-Schmidt norms, using (13) and Lemma 3.1, we have

$$\left\|M_0^{\pm}(\lambda)\right\|_{HS}^2 = \int_{\mathbb{R}^6} |v(x)\mathcal{E}_1^{\pm}(\lambda,x,y)v^*(y)|^2\,dx\,dy \lesssim |\lambda|^2 \int_{\mathbb{R}^6} \frac{\langle x\rangle^{-\delta}\langle y\rangle^{-\delta}}{|x-y|^2}\,dx\,dy \lesssim |\lambda|^2.$$

Lemma 3.6 was used in the $x$ integral, we use (16) with $p = 2$. Another application in the $y$ integral shows the integral is bounded. Then, applying Lemmas 3.1 and 3.6 show the bound. Similar computations show $\left\|\partial_\lambda M_0^{\pm}(\lambda)\right\|_{HS} \lesssim |\lambda|$ and $\left\|\partial_\lambda^2 M_0^{\pm}(\lambda)\right\|_{HS} < \infty$ requiring more decay from the potential to ensure that

$$\int_{\mathbb{R}^6} \langle x\rangle^{-\delta}\langle y\rangle^{-\delta}\,dx\,dy \qquad \text{and} \qquad \int_{\mathbb{R}^6} \langle x\rangle^{-\delta}|x-y|^2\langle y\rangle^{-\delta}\,dx\,dy$$

converge. Applying Lemma 3.6 requires $\delta > 3$ and $\delta > 5$, respectively. Now, we consider the first Lipschitz bound. For uniformity of presentation, we use that $|\lambda| \lesssim 1$ to obtain the bounds in the statement. By Lemma 3.2, we have

$$|M_0^{\pm}(\lambda_2) - M_0^{\pm}(\lambda_1)| \lesssim |\lambda_2 - \lambda_1|^{\gamma}|v(x)|\left(\frac{|\lambda_2|}{|x-y|^{1-\gamma}}\right)|v^*(y)|.$$

The Hilbert-Schmidt norm is bounded by Lemma 3.6, provided $\delta > 1 + 2\gamma$.

Finally for the derivative, again by Lemma 3.2, we have

$$|\partial_\lambda M_0^{\pm}(\lambda_2) - \partial_\lambda M_0^{\pm}(\lambda_1)| \lesssim |v(x)||\lambda_2 - \lambda_1|^{\gamma}|\lambda_2|^{1-\gamma}(1 + |x-y|^{\gamma})|v^*(y)|$$

This is Hilbert-Schmidt provided that $\delta > 3 + 2\gamma$.

Finally, note that $M^{\pm}(\lambda_2) - M^{\pm}(\lambda_1) = (\lambda_2 - \lambda_1)v\mathcal{G}_1 v^* + M_0^{\pm}(\lambda_2) - M_0^{\pm}(\lambda_1)$ since $T_0$ has no $\lambda$ dependence, while $\partial_\lambda M^{\pm}(\lambda_2) - \partial_\lambda M^{\pm}(\lambda_1) = \partial_\lambda \mathcal{E}_1^{\pm}(\lambda_2) - \partial_\lambda \mathcal{E}_1^{\pm}(\lambda_1)$ since $v\mathcal{G}_1 v^*$ is independent of $\lambda$. Using that $|\lambda_2 - \lambda_1| \lesssim |\lambda_2| \lesssim 1$, and the argument above for $M_0^{\pm}(\lambda)$ suffices to prove the claimed bounds for $M^{\pm}(\lambda)$. □

**Lemma 3.9.** *If zero is a regular point of the spectrum and* $|V(x)| \lesssim \langle x\rangle^{-\delta}$ *for some* $\delta > 1$*. Then* $M^{\pm}(\lambda)$ *is an invertible operator with uniformly bounded inverse on a sufficiently small interval* $0 < |\lambda| \ll 1$*. Furthermore,*

*i) If* $\delta > 1 + 2\gamma$ *for some* $0 \leq \gamma \leq 1$*, then for* $0 < |\lambda_1| \leq |\lambda_2| \ll 1$*, we have*

$$\|(M^{\pm})^{-1}(\lambda_2) - (M^{\pm})^{-1}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma}.$$

*ii) If* $\delta > 3$*, then*

$$\|\partial_\lambda (M^{\pm})^{-1}(\lambda)\|_{HS} < \infty. \tag{17}$$

*iii) If* $\delta > 3 + 2\gamma$ *for some* $0 \leq \gamma \leq 1$*, then for* $0 < |\lambda_1| \leq |\lambda_2| \ll 1$*, we have*

$$\|\partial_\lambda (M^{\pm})^{-1}(\lambda_2) - \partial_\lambda (M^{\pm})^{-1}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma}|\lambda_2|.$$

*Proof.* We consider the + case only and drop the superscript. By a Neumann series expansion, if $T_0$ is invertible on $L^2$, denote the inverse by $D_1 := T_0^{-1}$ as an operator on $L^2$. The fact that $D_1$ is an absolutely bounded operator is established in Lemma 2.10 of [23], which considered the massive operator. That proof applies here with only minimal modifications.

By Lemma 3.8 and a Neumann series expansion,

$$M^{-1}(\lambda) = (T_0 + \lambda v \mathcal{G}_1 v^* + M_0(\lambda))^{-1} = D_1(\mathbb{1} + \lambda v \mathcal{G}_1 v^* D_1 + M_0(\lambda)D_1)^{-1} = D_1 + O(|\lambda|),$$

where the error term is understood as a Hilbert-Schmidt operator. In particular, $M^{-1}(\lambda)$ is an absolutely bounded operator on $L^2(\mathbb{R}^3)$.

By the resolvent identity $M^{-1}(\lambda_1) - M^{-1}(\lambda_2) = M^{-1}(\lambda_1)(M(\lambda_1) - M(\lambda_2))M^{-1}(\lambda_2)$. Since $M$ was shown to be invertible, we may apply Lemma 3.8 to see

$$\begin{aligned}\big\|M^{-1}(\lambda_1) - M^{-1}(\lambda_2)\big\|_{HS} &\lesssim \|M^{-1}(\lambda_1)\|_{L^2\to L^2}\|(M(\lambda_1) - M(\lambda_2))\|_{HS}\|M^{-1}(\lambda_2)\|_{L^2\to L^2}\\ &\lesssim |\lambda_2 - \lambda_1|^{\gamma},\end{aligned}$$

provided $1 + 2\gamma > \delta$. For the second claim, (17), we use the identity

$$\partial_\lambda M^{-1}(\lambda) = -M^{-1}(\lambda)(\partial_\lambda M(\lambda))M^{-1}(\lambda) \tag{18}$$

By Lemma 3.8 and the boundedness of $M^{-1}$ established above, we see

$$\|\partial_\lambda M^{-1}(\lambda)\|_{HS} \lesssim \|\partial_\lambda M(\lambda)\|_{HS} \lesssim \lambda < \infty.$$

Finally, for the Lipschitz bound we recall the following useful algebraic identity

$$\prod_{k=0}^{M} A_k(\lambda_2) - \prod_{k=0}^{M} A_k(\lambda_1) = \sum_{\ell=0}^{M}\Big(\prod_{k=0}^{\ell-1} A_k(\lambda_1)\Big)\big(A_\ell(\lambda_2) - A_\ell(\lambda_1)\big)\Big(\prod_{k=\ell+1}^{M} A_k(\lambda_2)\Big). \tag{19}$$

The algebraic identity ensures that there is a difference of the same operators evaluated at $\lambda_2$ and $\lambda_1$ on which we may apply the previously obtained Lipschitz bounds. By (19) and (18)

$$\partial_\lambda M^{-1}(\lambda_2) - \partial_\lambda M^{-1}(\lambda_1) = [M^{-1}(\lambda_2) - M^{-1}(\lambda_1)]\partial_\lambda(M(\lambda_2))M^{-1}(\lambda_2)$$
$$+ M^{-1}(\lambda_1)[\partial_\lambda M^{-1}(\lambda_2) - \partial_\lambda M^{-1}(\lambda_1)]M^{-1}(\lambda_2) + M^{-1}(\lambda_1)\partial_\lambda(M(\lambda_1))[M^{-1}(\lambda_2) - M^{-1}(\lambda_1)].$$

Since $\partial_\lambda M^{-1}(\lambda)$ and $M^{-1}(\lambda)$ are absolutely bounded, by Lemma 3.8, we see

$$\|\partial_\lambda M^{-1}(\lambda_1) - \partial_\lambda M^{-1}(\lambda_2)\|_{HS} \lesssim |\lambda_1 - \lambda_2|^\gamma |\lambda_2|.$$

Here, we need $\delta > 3 + 2\gamma$ to apply the Lipschitz bound of Lemma 3.8. □

## 4. Dispersive bounds when zero is regular

Now that we have developed appropriate expansions for the free and perturbed resolvent operators, we are ready to prove the low energy claims in Theorem 1.1 in the case when zero is regular. Using the differentiability properties established in Section 3, we develop expansions of the spectral measure in a neighborhood of zero with similar properties. Using the Stone's formula, (6), we reduce the evolution bounds to oscillatory integrals that we control. We first prove the uniform, $L^1 \to L^\infty$, bounds. Then, in Subsection 4.1 we utilize the more delicate Lipschitz continuity bounds to prove the large time-integrable bounds at the cost of mapping between weighted spaces. Finally, we apply the Lipschitz bounds to prove the low energy version of Theorem 4.7.

By iterating the symmetric resolvent identity, (10), we obtain the Born series expansion:

$$\mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda)v^*(M^\pm)^{-1}v\mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda). \tag{20}$$

We iterate so that we have two resolvents on either side of $(M^\pm)^{-1}$ since the kernel of $\mathcal{R}_0^\pm(\lambda)$ has a leading term that is not locally $L^2(\mathbb{R}^3)$. The main goal of this section is to use the Born series expansion to prove low energy dispersive bounds.

**Proposition 4.1.** *Assume that $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 3$. Then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)(\mathcal{R}_V^+ - \mathcal{R}_V^-)(\lambda)(x,y)\,d\lambda\right| \lesssim \langle t\rangle^{-1}.$$

We prove this Proposition through a series of Lemmas. First, we control the contribution of the Born series, the first three terms in (20), to the Stone's formula. We note that the first term is the free evolution and is controlled in Theorem 2.2.

**Lemma 4.2.** *Assume $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for $\delta > 2$. Then for any fixed $k \in \mathbb{N}$, we have the bound*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)\Big(\mathcal{R}_0^+(\lambda)(V\mathcal{R}_0^+(\lambda))^k - \mathcal{R}_0^-(\lambda)(V\mathcal{R}_0^-(\lambda))^k\Big)d\lambda\right| \lesssim \frac{1}{|t|}.$$

*Proof.* To utilize the difference between the '+' and '−' resolvent operators, we use the following algebraic identity.

$$\prod_{k=0}^{M} A_k^+ - \prod_{k=0}^{M} A_k^- = \sum_{\ell=0}^{M} \Big( \prod_{k=0}^{\ell-1} A_k^- \Big) (A_\ell^+ - A_\ell^-) \Big( \prod_{k=\ell+1}^{M} A_k^+ \Big). \tag{21}$$

We first prove the claim when $k = 1$. Integrating by parts and expanding the derivative for the $k = 1$ case yields four terms, the first of which we will bound since the other three follow similarly. Recall that $\mu_0(\lambda, x, y) = \chi(\lambda)[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda, x, y)$. Using (12) and (7) we integrate by parts once to see

$$\Big| \int_{\mathbb{R}} \frac{e^{-it\lambda}}{t} \int_{\mathbb{R}^3} \partial_\lambda \big( \mu_0(\lambda)(x, z) V(z) \mathcal{R}_0^\pm(\lambda, z, y) \big) dz d\lambda \Big|$$
$$\lesssim \frac{1}{|t|} \int_{-1}^{1} \int_{\mathbb{R}^3} \langle z \rangle^{-\delta} \Big( \frac{1}{|z-y|^2} + \frac{1}{|z-y|} + \frac{1}{|x-z||z-y|} + \frac{1}{|z-y|} \Big) dz\, d\lambda \lesssim \frac{1}{|t|},$$

here we use Lemma 3.6 since $\delta > 2$. This bound is uniform in $x$ and $y$. The boundedness of the integral for small $t$ follows without integrating by parts but similarly using (7), (12) and Lemma 3.6, to control the spatial integrals.

When $k > 1$ we note that applying Lemma 3.6 with $\delta > 2$, and (12), on the support of $\chi(\lambda)$ we have

$$\Big| \int_{\mathbb{R}^3} \mathcal{R}_0^\pm(\lambda)(x, z_1) V(z_1) \mathcal{R}_0^\pm(z_1, z_2)\, dz_1 \Big| \lesssim \frac{1}{|x - z_2|^2} + \frac{1}{|x - z_2|},$$

which is the same upper bound we use in these arguments for $\mathcal{R}_0^\pm(\lambda)(x, z_2)$. Hence one may reduce to the argument when $k = 1$ by first integrating in the spatial variables of resolvents that are not differentiated to reduce to bounding spatial integrals of the form considered above. □

**Lemma 4.3.** *If* $|V(x)| \lesssim \langle x \rangle^{-\delta}$ *for any* $\delta > 3$ *and* $\Gamma(\lambda)$ *is a absolutely bounded operator satisfying*

$$\big\| |\Gamma(\lambda)| \big\|_{L^2 \to L^2} + \big\| |\lambda \partial_\lambda \Gamma(\lambda)| \big\|_{L^2 \to L^2} \lesssim |\lambda|^{0+},$$

*then we have the bound*

$$\sup_{x,y \in \mathbb{R}^3} \Big| \int_{\mathbb{R}} e^{-it\lambda} \chi(\lambda) \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda) v^* \Gamma(\lambda) v \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda)\, d\lambda \Big| \lesssim \langle t \rangle^{-1}.$$

The assumptions on the operator $\Gamma(\lambda)$ are far less stringent than needed here since $\partial_\lambda (M^\pm)^{-1}(\lambda)$ is bounded in a neighborhood of zero when zero is regular. We choose to prove a more general result to reuse in the analysis when zero is not regular.

*Proof.* As before we integrate by parts once to bound

$$\frac{1}{|t|} \sup_{x,y \in \mathbb{R}^3} \Big| \int_{\mathbb{R}} e^{-it\lambda} \partial_\lambda \big( \chi(\lambda) \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda) v^* \Gamma(\lambda) v \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda) \big)\, d\lambda \Big|. \tag{22}$$

The assumptions on $\Gamma(\lambda)$ and (12) ensure there are no boundary terms at zero. Furthermore, when $|\lambda| \ll 1$, by (12) we have

$$\big|(\mathcal{R}_0^{\pm} V \mathcal{R}_0^{\pm})(\lambda)(z_2,x)\big| \lesssim \int_{\mathbb{R}^3} \langle z_1\rangle^{-\delta} \left( \frac{1}{|z_2-z_1|^2\,|z_1-x|^2} + \frac{1}{|z_2-z_1|\,|z_1-x|}\right) dz_1 \lesssim \frac{1}{|z_2-x|},$$

by Lemma 3.6 provided $\delta > 2$. Using Lemma 3.6 again in the $z_2$ integral we see that

$$\sup_{x\in\mathbb{R}^3} \big\|(v\mathcal{R}_0^{\pm} V \mathcal{R}_0^{\pm})(\lambda)(\,\cdot\,,x)\big\|_{L^2}^2 \lesssim 1. \tag{23}$$

By duality, $\big\|\big(\mathcal{R}_0^{\pm} V \mathcal{R}_0^{\pm} v^*\big)(\lambda)(x,\,\cdot\,)\big\|_{L^2} \lesssim 1$ holds uniformly in $x$ as well. Using (12) and Lemma 3.6, we see

$$\sup_{x\in\mathbb{R}^3} \big\|\partial_\lambda\big(v\mathcal{R}_0^{\pm} V \mathcal{R}_0^{\pm}\big)(\lambda)(x,\,\cdot\,)\big\|_{L^2}^2 \lesssim 1. \tag{24}$$

One uses that $\delta > 3$ here to ensure that the contribution the slowest decaying terms after the applying Lemma 3.6 are in $L^2$. Bringing everything together, we may express the contribution of (22) by rewriting the integral as

$$(22) = \frac{1}{t}\int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}(\lambda)\,d\lambda,$$

where

$$\mathcal{E}(\lambda) = \sum \big(\partial_\lambda^{k_1}\chi(\lambda)\big)\partial_\lambda^{k_2}\big(\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)v^*\big)\big(\partial_\lambda^{k_3}\Gamma(\lambda)\big)\partial_\lambda^{k_4}\big(v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)\big),$$

and the sum is taken over the indices $k_j \in \{0,1\}$ subject to $k_1+k_2+k_3+k_4 = 1$. By the absolute boundedness of $\Gamma(\lambda)$, we have

$$\begin{aligned}
|\mathcal{E}(\lambda)| &= \big|\big(\partial_\lambda^{k_1}\chi(\lambda)\big)\partial_\lambda^{k_2}\big(\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)v^*\big)\big(\partial_\lambda^{k_3}\Gamma(\lambda)\big)\partial_\lambda^{k_4}\big(v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)\big)\big| \\
&= \Big\langle \partial_\lambda^{k_2}\big(v\mathcal{R}_0^{\mp}(\lambda)V\mathcal{R}_0^{\mp}(\lambda)\big), \big(\partial_\lambda^{k_3}\Gamma(\lambda)\big)\partial_\lambda^{k_4}\big(v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)\big)\Big\rangle_{L^2} \\
&\lesssim \big\|\partial_\lambda^{k_2}\big((v\mathcal{R}_0^{\pm}V\mathcal{R}_0^{\pm})(\,\cdot\,,x)\big)\big\|_{L^2}\big\||\partial_\lambda^{k_3}\Gamma(\lambda,z_2,\,\cdot\,)|\partial_\lambda^{k_4}\big((\mathcal{R}_0^{\pm}V\mathcal{R}_0^{\pm}v^*)(\,\cdot\,,y)\big)\big\|_{L^2} \\
&\lesssim \big\|\partial_\lambda^{k_2}\big((v\mathcal{R}_0^{\pm}V\mathcal{R}_0^{\pm})(\,\cdot\,,x)\big)\big\|_{L^2}\big\||\partial_\lambda^{k_3}\Gamma(\lambda)|\big\|_{L^2\to L^2}\big\|\partial_\lambda^{k_4}\big((\mathcal{R}_0^{\pm}V\mathcal{R}_0^{\pm}v^*)(\,\cdot\,,y)\big)\big\|_{L^2} \lesssim |\lambda|^{-1+},
\end{aligned}$$

which holds over the support of $\chi$, which is contained in the interval $[-1,1]$. Applying this bound to (22), we have

$$(22) \lesssim \frac{1}{|t|}\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}(\lambda)\,d\lambda\right| \lesssim \frac{1}{|t|}\sup_{x,y\in\mathbb{R}^3}\int_{-1}^{1}|\lambda|^{-1+}\,d\lambda \lesssim \frac{1}{|t|}$$

as desired. The claim for boundedness for small $|t|$ follows the argument for when $k_1 = 1$ without integrating by parts. □

Now, we prove Proposition 4.1.

*Proof of Proposition 4.1.* By expanding $\mathcal{R}_V^{\pm}$ into a Born series expansion as in (20), we can control the contribution of each term. The contribution of first term in (20) to (4) is controlled by Theorem 2.2, the contribution of the second and third are controlled by Lemma 4.2. For the final term, we do not utilize the difference between the '+' and '−' resolvent, but control each by applying Lemma 4.3. □

4.1. **Weighted dispersive bounds when zero is regular.** We now turn to showing that the large time integrable bounds hold when zero is regular. Here we utilize the Lipschitz continuity of the perturbed resolvent and its first derivative in a neighborhood of the threshold at $\lambda = 0$. Our main goal is to show the bound

**Proposition 4.4.** *Fix $0 < \gamma \leq 1$. If $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 3 + 2\gamma$, then for $|t| > 1$ we have the weighted bound*

$$\bigg| \int_{\mathbb{R}} e^{-it\lambda} \chi(\lambda) [\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)(x, y)\, d\lambda \bigg| \lesssim \frac{\langle x\rangle^{\gamma} \langle y\rangle^{\gamma}}{|t|^{1+\gamma}}.$$

This tells us that the low energy portion of the evolution satisfies a large time-integrable bound as an operator from $L^{1,\gamma} \to L^{\infty,-\gamma}$. As in the proof of the uniform bound in the previous subsection, we consider the Born series expansion (20) and bound each term individually.

**Lemma 4.5.** *Under the assumptions of Proposition 4.4, for any fixed $k \in \mathbb{N}$ we have*

$$\bigg| \int_{\mathbb{R}} e^{-it\lambda} \chi(\lambda) \Big( \mathcal{R}_0^+(\lambda)(V\mathcal{R}_0^+(\lambda))^k - \mathcal{R}_0^-(\lambda)(V\mathcal{R}_0^-(\lambda))^k \Big)\, d\lambda \bigg| \lesssim \frac{\langle x\rangle^{\gamma} \langle y\rangle^{\gamma}}{|t|^{1+\gamma}}.$$

*Proof.* We apply Lemma 2.1 and the Lipschitz bounds on the resolvents using $\lambda_2 = \lambda$ and $\lambda_1 = \lambda - \pi/t$ for large $|t|$, so that $|\lambda_2 - \lambda_1| = \pi/|t|$ is small. We first consider the case when $k = 1$. By (21), as in the proof of Lemma 4.2, it suffices to control

$$\int_{\mathbb{R}} e^{-it\lambda} \mu_0(\lambda) V \mathcal{R}_0^+(\lambda)\, d\lambda = \frac{1}{it} \int_{\mathbb{R}} e^{-it\lambda} \big( \partial_\lambda [\mu_0 V \mathcal{R}_0^+](\lambda) - \partial_\lambda [\mu_0 V \mathcal{R}_0^+](\lambda - \pi/t) \big)\, d\lambda. \tag{25}$$

We first consider the contribution of $\partial_\lambda \mu_0 V \mathcal{R}_0^+$. To utilize the Lipschitz bounds we apply (19) to see

$$[\partial_\lambda \mu_0(\lambda) - \partial_\lambda \mu_0(\lambda - \pi/t)] V \mathcal{R}_0^+(\lambda) + \partial_\lambda \mu_0(\lambda - \pi/t) V [\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^+(\lambda - \pi/t)]. \tag{26}$$

Applying (7), (9), (12), Corollary 3.3, and including the spatial variable dependence, we see that

$$|(26)| \lesssim |t|^{-\gamma} |x - z|^{\gamma} |\lambda| |V(z)| \bigg( \frac{1}{|z - y|^2} + \frac{|\lambda|}{|z - y|} + \frac{|\lambda|}{|z - y|^{1-\gamma}} \bigg) + |t|^{-1} |V(z)| \frac{|\lambda|^2}{|z - y|}.$$

Since $|t| > 1$, $|\lambda| \lesssim 1$, and $|x - z|^{\gamma} \lesssim \langle x\rangle^{\gamma} \langle z\rangle^{\gamma}$,

$$|(26)| \lesssim |t|^{-\gamma} \langle x\rangle^{\gamma} \langle z\rangle^{\gamma-\delta} \bigg( \frac{1}{|z - y|^2} + \frac{1}{|z - y|^{1-\gamma}} \bigg).$$

Applying Lemma 3.6 to control the spatial integrals, along with the support of $\chi$ shows that the contribution of (26) to (25) is bounded by $|t|^{-1-\gamma} \langle x\rangle^{\gamma}$. In the case that the $\lambda$ derivative acts on the resolvent on the right, applying (7) and Corollary 3.3 shows that its contribution to (25) may be bounded by

$$\frac{1}{|t|^{1+\gamma}}\int_{\mathbb{R}^3}\langle z\rangle^{\gamma-\delta}\bigg(\frac{1}{|z-y|}+\langle y\rangle^{\gamma}\bigg)\,dz\lesssim\frac{\langle y\rangle^{\gamma}}{|t|^{1+\gamma}}.$$

A similar analysis shows that

$$\big|\mu_0(\lambda_1)V[\partial_\lambda\mathcal{R}_0(\lambda_2)-\partial_\lambda\mathcal{R}_0(\lambda_1)]\big|\lesssim|t|^{-\gamma}\langle y\rangle^{\gamma}\langle z\rangle^{\gamma-\delta}\bigg(\frac{1}{|z-x|^2}+\frac{1}{|z-x|^{1-\gamma}}\bigg).$$

Applying Lemma 3.6 completes the proof provided $\delta>3+2\gamma$.

For $k>1$, similar to the proof of Lemma 4.2, we may iterate this argument and apply Lemma 3.6 to see that the iterated spatial integrals are effectively harmless. Using (21), we see that we may use the Lipschitz bounds on one resolvent in the product, while the remaining resolvents are all bounded as before. Applying the bounds (7), (8), (12) and Corollary 3.3 repeatedly suffices to prove the claim. □

We now turn to the tail of the Born series. We do not rely on the difference between the '+' and '−' resolvents here, since we showed the iterated resolvents are locally $L^2$ in the proof of Lemma 4.3, we bound both the '+' and '−' resolvent contributions in one step. As before, the assumptions on $\Gamma(\lambda)$ are less stringent than needed in the case when zero is regular.

**Lemma 4.6.** *Fix $0<\gamma\leq1$, and suppose that $\Gamma(\lambda)$ and $\partial_\lambda\Gamma(\lambda)$ are absolutely bounded operators satisfying*

$$\big\||\Gamma(\lambda)|\big\|_{L^2\to L^2}+\big\||\lambda\partial_\lambda\Gamma(\lambda)|\big\|_{L^2\to L^2}\lesssim|\lambda|^{0+},$$

*and the Lipschitz bounds*

$$\big\||\Gamma(\lambda_2)-\Gamma(\lambda_1)|\big\|_{L^2\to L^2}+\big\||\partial_\lambda\Gamma(\lambda_2)-\partial_\lambda\Gamma(\lambda_1)|\big\|_{L^2\to L^2}\lesssim|\lambda_2-\lambda_1|^{\gamma}|\lambda_1|^{-1+},$$

*when $0<|\lambda_1|\leq|\lambda_2|\leq1$. If $|V(x)|\lesssim\langle x\rangle^{-\delta}$ for some $\delta>3+2\gamma$, then for $|t|>1$, we have the bound*

$$\bigg|\int_{\mathbb{R}}e^{-it\lambda}\chi(\lambda)\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)v^*\Gamma(\lambda)v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)\,d\lambda\bigg|\lesssim\frac{\langle x\rangle^{\gamma}\langle y\rangle^{\gamma}}{|t|^{1+\gamma}}.$$

*Proof.* Again, we reduce this to an application of Lemma 2.1 and the Lipschitz bounds for the resolvents and $\Gamma(\lambda)$. We consider only the '+' case, the '−' follows identically. The first assumptions on $\Gamma$ and (12) ensure there are no boundary terms at zero when integrating by parts. After one application of integration by parts, we have two cases to consider. Either the derivative acts on a resolvent, or it acts on $\Gamma(\lambda)$. Consider the first case, here we note that it suffices to consider

$$\bigg|\int_{\mathbb{R}}e^{-it\lambda}\chi(\lambda)\partial_\lambda[\mathcal{R}_0^{+}(\lambda)V\mathcal{R}_0^{+}(\lambda)]v^*\Gamma(\lambda)v\mathcal{R}_0^{+}(\lambda)V\mathcal{R}_0^{+}(\lambda)\,d\lambda\bigg|. \tag{27}$$

Noting the Lipschitz bounds in Lemma 3.9, it suffices to show that the iterated resolvent and its derivative satisfy appropriate Lipschitz bounds. Applying (21), we see that

$$(\partial_\lambda^j\mathcal{R}_0^+)V\mathcal{R}_0^+(\lambda_2)-(\partial_\lambda^j\mathcal{R}_0^+)V\mathcal{R}_0^+(\lambda_1) = [\partial_\lambda^j\mathcal{R}_0^+(\lambda_2)-\partial_\lambda^j\mathcal{R}_0^+(\lambda_1)]V\mathcal{R}_0^+(\lambda_2)+\partial_\lambda^j\mathcal{R}_0^+(\lambda_1)V[\mathcal{R}_0^+(\lambda_2)-\mathcal{R}_0^+(\lambda_1)]. \tag{28}$$

When $j=0$ applying the bounds in (12), Corollary 3.3 and Lemma 3.6, to $\lambda_2=\lambda$ and $\lambda_1=\lambda-\pi/t$ when $|t|>1$ we see that

$$|(28)|\lesssim |t|^{-\gamma}\int_{\mathbb{R}^3}\frac{1+|x-z_1|^\gamma}{|x-z_1|}\langle z_1\rangle^{-\delta}\frac{1+|x-y|}{|x-y|^2}\,dz_1\lesssim |t|^{-\gamma}|x-y|^{-1}.$$

To apply Lemma 3.6 when $k+\ell=3$, we used the crude bound $a^{-1}b^{-2}\lesssim a^{-2}b^{-2}+a^{-1}b^{-1}$ for $a,b>0$. Applying a similar argument when $j=1$ results in a weight in $x$, namely the first summand contributes

$$|[\partial_\lambda\mathcal{R}_0^+(\lambda_2)-\partial_\lambda\mathcal{R}_0^+(\lambda_1)]V\mathcal{R}_0^+(\lambda_2)|\lesssim |t|^{-\gamma}\int_{\mathbb{R}^3}(1+|x-z_1|^\gamma)\langle z_1\rangle^{-\delta}\frac{1+|z_1-y|}{|z_1-y|^2}\,dz_1\lesssim |t|^{-\gamma}\langle x\rangle^\gamma.$$

From this, through an application of Lemma 3.7, we can see the Lipschitz bounds of the $L^2$ norms of iterated resolvents

$$\|v(\,\cdot\,)\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_2)-v(\,\cdot\,)\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_1)\|_{L^2}\lesssim |t|^{-\gamma}, \tag{29}$$

$$\|v(\,\cdot\,)\big(\partial_\lambda[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_2)](\,\cdot\,,y)-\partial_\lambda[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_1)](\,\cdot\,,y)\big)\|_{L^2}\lesssim |t|^{-\gamma}\langle y\rangle^\gamma. \tag{30}$$

Noting that the spatial integrals that arise in applying Lemma 2.1 and (21) to (27) may be controlled by a sum of terms of the form:

$$\begin{aligned}
&\big\|\big(\partial_\lambda^{j_1}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_2)](x,\,\cdot\,)-\partial_\lambda^{j_1}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_1)](\,\cdot\,,x)\big)v^*(\,\cdot\,)\big\|_{L^2}\\
&\qquad\times\big\||\partial_\lambda^{j_2}\Gamma(\lambda)|\big\|_{L^2\to L^2}\big\|v(\,\cdot\,)\partial_\lambda^{j_3}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda)]\big\|_{L^2}\\
&+\big\|\partial_\lambda^{j_1}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda)](x,\,\cdot\,)v^*(\,\cdot\,)\big\|_2\big\||\partial_\lambda^{j_2}\big(\Gamma(\lambda_2)-\Gamma(\lambda_1)\big)|\big\|_{L^2\to L^2}\big\|v(\,\cdot\,)\partial_\lambda^{j_3}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda)]\big\|_{L^2}\\
&+\big\|\partial_\lambda^{j_1}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda)](x,\,\cdot\,)v^*(\,\cdot\,)\big\|_{L^2}\big\||\partial_\lambda^{j_2}\Gamma(\lambda)|\big\|_{L^2\to L^2}\big\|v(\,\cdot\,)\partial_\lambda^{j_3}[\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_2)-\mathcal{R}_0^+V\mathcal{R}_0^+(\lambda_1)]\|_{L^2},
\end{aligned}$$

where $j_1,j_2,j_3\in\{0,1\}$ and $j_1+j_2+j_3=1$. Combining equations (29), (30), the assumptions on $\Gamma$, and the support of $\chi$ show that

$$|(27)|\lesssim\left|\int_{\mathbb{R}}e^{-it\lambda}\mathcal{E}(\lambda)\,d\lambda\right|,$$

where $\mathcal{E}(\lambda)$ is supported on $(-1,1)$ and $\mathcal{E}'(\lambda)$ is integrable with

$$|\mathcal{E}'(\lambda)-\mathcal{E}'(\lambda-\pi/t)|\lesssim |t|^{-\gamma}\langle x\rangle^\gamma\langle y\rangle^\gamma|\lambda|^{-1+}.$$

Applying Lemma 2.1 proves the claim. □

We note here that the Lipschitz bounds used for the resolvents in the proof, Lemma 3.2 and Corollary 3.3, the extra smallness in $\lambda$ was not used. That is, we dominate all positive powers of $|\lambda_2|$ by a constant in this proof. We are now ready to prove Proposition 4.4.

*Proof of Proposition 4.4.* By expanding $\mathcal{R}_V^\pm$ into a Born series expansion as in (20), we can control the contribution of each term. The contribution of first term in (20) to (4) is controlled by Theorem 2.2, the contribution of the second and third are controlled by Lemma 4.5. For the final term, we do not utilize the difference between the '+' and '−' resolvent, but control each by applying Lemma 4.6. □

We note that one can apply the Lipschitz bounds directly without integrating by parts to prove weaker versions of this theorem that require less decay on the potential as in Theorem 1.3. Namely,

**Theorem 4.7.** *Fix a value of $0 \leq \gamma \leq 1$. If zero is regular and $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 1 + 2\gamma$, then*

$$\|e^{-it\mathcal{H}}P_{ac}(\mathcal{H})\chi(\mathcal{H})\|_{L^1\to L^\infty} \lesssim \langle t\rangle^{-\gamma}.$$

This shows, for weaker decay on the potential, that the low energy portion of the evolution may be controlled. In particular, the evolution is bounded if $\delta > 1$.

*Proof.* Instead of applying Lemma 2.1 as in the proofs of the previous theorems, if $\mathcal{E}(\lambda)$ is bounded we instead apply

$$\left| \int_{\mathbb{R}} e^{-it\lambda}\mathcal{E}(\lambda)\, d\lambda \right| \lesssim \int_{\mathbb{R}} \left| \mathcal{E}(\lambda) - \mathcal{E}\left(\lambda - \frac{\pi}{t}\right)\right| d\lambda$$

to the Stone's formula, (6). This follows from the proof of Lemma 2.1 without integrating by parts first.

Here, instead of iterating the resolvent identity directly, we note that we may write

$$\begin{aligned} \mathcal{R}_0^\pm(\lambda)(x,y) &= \chi(\lambda|x-y|)\mathcal{R}_0^\pm(\lambda)(x,y) + \widetilde{\chi}(\lambda|x-y|)\mathcal{R}_0^\pm(\lambda)(x,y) \\ &:= \mathcal{R}_L^\pm(\lambda)(x,y) + \mathcal{R}_H^\pm(\lambda)(x,y). \end{aligned} \tag{31}$$

Here $\widetilde{\chi} = 1 - \chi$ is a smooth cut-off away from a neighborhood of zero. By the expansions developed in Lemma 3.1, we have (for $k = 0, 1$)

$$|\partial_\lambda^k \mathcal{R}_L^\pm(\lambda)(x,y)| \lesssim |x-y|^{k-2}, \qquad |\partial_\lambda^k \mathcal{R}_H^\pm(\lambda)(x,y)| \lesssim |\lambda|\,|x-y|^{k-1}. \tag{32}$$

In particular, we note that $\mathcal{R}_H^\pm$ is a locally $L^2$ function of $x$ or $y$. As before, we may use these bounds to obtain Lipschitz bounds (for $|\lambda_1| \leq |\lambda_2| \leq 1$ and any $0 \leq \gamma \leq 1$)

$$|\mathcal{R}_L^\pm(\lambda_2)(x,y) - \mathcal{R}_L^\pm(\lambda_1)(x,y)| \lesssim \frac{|\lambda_2-\lambda_1|^\gamma}{|x-y|^{2-\gamma}}, \tag{33}$$

$$|\mathcal{R}_H^{\pm}(\lambda_2)(x,y) - \mathcal{R}_H^{\pm}(\lambda_1)(x,y)| \lesssim \frac{|\lambda_2|\,|\lambda_2-\lambda_1|^{\gamma}}{|x-y|^{1-\gamma}}. \tag{34}$$

From here we may selectively iterate the symmetry resolvent identity:

$$\mathcal{R}_V^{\pm}(\lambda)V = \mathcal{R}_0^{\pm}(\lambda)v^*(M^{\pm})^{-1}(\lambda)v$$

to form a Born series expansion tailored to optimize the decay needed from the potential.

$$\begin{aligned}(35)\quad \mathcal{R}_V^{\pm}(\lambda) &= \mathcal{R}_0^{\pm}(\lambda) - \mathcal{R}_H^{\pm}(\lambda)v^*(M^{\pm})^{-1}(\lambda)v\mathcal{R}_H^{\pm}(\lambda) - \mathcal{R}_L^{\pm}(\lambda)V\mathcal{R}_0 v^*(M^{\pm})^{-1}(\lambda)v\mathcal{R}_H^{\pm}(\lambda)\\ &- \mathcal{R}_H^{\pm}(\lambda)v^*(M^{\pm})^{-1}(\lambda)v\mathcal{R}_0^{\pm}V\mathcal{R}_L^{\pm}(\lambda) + \mathcal{R}_L^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)v^*(M^{\pm})^{-1}(\lambda)v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_L^{\pm}(\lambda).\end{aligned}$$

Using (31) and Lemma 3.7 $\mathcal{R}_H^{\pm}(\lambda)(x,\cdot)v^*(\cdot)$ is in $L^2$ uniformly in $x$ provided $\delta > 1/2$. On the other hand, using (34) and Lemma 3.7 shows that

$$\|\mathcal{R}_H^{\pm}(\lambda_2)(x,\cdot)v^*(\cdot) - \mathcal{R}_H^{\pm}(\lambda_1)(x,\cdot)v^*(\cdot)\|_2 \lesssim |\lambda_2-\lambda_1|^{\gamma}|\lambda_2|$$

uniformly in $x$ provided $\delta > \gamma + 1/2$. Applying (19) we see that

$$\begin{aligned}&\mathcal{R}_L^{\pm}(\lambda_2)V\mathcal{R}_0(\lambda_2)v^* - \mathcal{R}_L^{\pm}(\lambda_1)V\mathcal{R}_0(\lambda_1)v^*\\ &\qquad = [\mathcal{R}_L^{\pm}(\lambda_2) - \mathcal{R}_L^{\pm}(\lambda_1)]V\mathcal{R}_0^{\pm}(\lambda_2)v^* + \mathcal{R}_L^{\pm}(\lambda_1)V[\mathcal{R}_0(\lambda_2) - \mathcal{R}_0(\lambda_2)]v^*\end{aligned}$$

By (12), (32), (33), and Corollary 3.3 we see that the size of the integral kernel of this operator is bounded by

$$|\lambda_2-\lambda_1|^{\gamma}|v^*(z_2)|\int_{\mathbb{R}^3}|V(z_1)|\left(\frac{1+|z_1-z_2|}{|x-z_1|^{2-\gamma}|z_1-z_2|^2} + \frac{1+|z_1-z_2|^{\gamma}}{|x-z_1|^2|z_1-z_2|}\right)dz_1.$$

Applying Lemma 3.6, we note that the integration smooths out the local singularity enough to be locally $L^2$. The decay of the resulting upper bound in terms of $x, z_2$ is constrained by the case when $k=2$ and $\ell = 1-\gamma$. From this we see that if $\delta > 1+\gamma$ we have the upper bound

$$\big|[\mathcal{R}_L^{\pm}(\lambda_2)V\mathcal{R}_0(\lambda_2)v^* - \mathcal{R}_L^{\pm}(\lambda_1)V\mathcal{R}_0(\lambda_1)v^*](x,z_2)\big| \lesssim |\lambda_2-\lambda_1|^{\gamma}\frac{\langle z_2\rangle^{-\frac{1+\gamma}{2}-}}{|x-z_2|^{1-\gamma}}.$$

Applying Lemma 3.7 we see

$$\sup_{x\in\mathbb{R}^3}\|[\mathcal{R}_L^{\pm}(\lambda_2)V\mathcal{R}_0(\lambda_2)(x,\cdot) - \mathcal{R}_L^{\pm}(\lambda_1)V\mathcal{R}_0(\lambda_1)(x,\cdot)]v^*(\cdot)\|_{L^2} \lesssim |\lambda_2-\lambda_1|^{\gamma}$$

A similar analysis shows that if $\delta > 1$, then

$$\sup_{x\in\mathbb{R}^3}\|\mathcal{R}_L^{\pm}(\lambda)V\mathcal{R}_0(\lambda)(x,\cdot)\|_{L^2} \lesssim 1.$$

We further require $\delta > 1+2\gamma$ to obtain the Lipschitz bounds on $(M^{\pm})^{-1}(\lambda)$ in Lemma 3.9.

The claim now follows by selecting $\lambda_2 = \lambda$ and $\lambda_1 = \lambda - \pi/t$ for $|t| > 1$, here $\mathcal{E}(\lambda) = \chi(\lambda)[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)$. The spatial integrals are controlled by the $L^2$ norms found above with an analysis similar to that in the proof of Lemma 4.3. □

## 5. Dispersive bounds when zero is not regular

We now consider the low energy evolution when zero energy is not regular. As shown in Section 6 below, if zero energy is not regular the operator $\mathcal{H}$ has a zero energy eigenvalue. This further complicates the inversion process near the threshold, and results in an expansion for the spectral measure that is singular as $\lambda \to 0$. We show that this singularity may be overcome, with only a slight increase in the needed decay on $V$. We show this by utilizing the cancellation between the '+' and '−' resolvents that was not needed in the regular case to overcome the loss of powers of $\lambda$ that arise due to the presence of a zero energy eigenvalue.

The main result of this section are the dispersive bounds when zero is not regular.

**Proposition 5.1.** *Assume that $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 3$. If zero is not a regular point of $\mathcal{H}$, then*

$$\sup_{x,y\in\mathbb{R}^3} \left| \int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)(\mathcal{R}_V^+ - \mathcal{R}_V^-)(\lambda)(x,y)\, d\lambda \right| \lesssim \langle t\rangle^{-1}.$$

*Further, for fixed $0 \leq \gamma < \frac{1}{2}$, if $\delta > 3 + 4\gamma$, then for $|t| > 1$ we have the weighted bound*

$$\sup_{x,y\in\mathbb{R}^3} \left| \int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)(\mathcal{R}_V^+ - \mathcal{R}_V^-)(\lambda)(x,y)\, d\lambda \right| \lesssim \frac{\langle x\rangle^{\gamma}\langle y\rangle^{\gamma}}{\langle t\rangle^{1+\gamma}}.$$

These dispersive bounds follow by developing an appropriate expansion for the operators $(M^{\pm}(\lambda))^{-1}$ that account for the existence of the zero energy eigenvalues in Proposition 5.6 below.

To invert $M^{\pm}(\lambda) = U + v\mathcal{R}_0^{\pm}(\lambda^2)v^*$, for small $\lambda$, we use the following Lemma (see Lemma 2.1 in [33]) with $S = S_1$, the Riesz projection onto the kernel of $T_0 = M^{\pm}(0) = U + v\mathcal{G}_0 v^*$.

**Lemma 5.2.** *Let $M$ be a closed operator on a Hilbert space $\mathcal{H}$ and $S$ a projection. Suppose $M + S$ has a bounded inverse. Then $M$ has a bounded inverse if and only if*

$$B := S - S(M + S)^{-1}S$$

*has a bounded inverse in $S\mathcal{H}$, and in this case*

$$M^{-1} = (M + S)^{-1} + (M + S)^{-1}SB^{-1}S(M + S)^{-1}.$$

Here, we have

$$B^{\pm}(\lambda) = S_1 - S_1(M^{\pm}(\lambda) + S_1)^{-1}S_1.$$

We note that, with a slight abuse of notation, the expansions for $(M^{\pm})^{-1}(\lambda)$ in Lemma 3.9 all hold for $(M^{\pm}(\lambda) + S_1)^{-1}$ with $D_1 = (T_0 + S_1)^{-1}$. When zero is regular, $S_1 = 0$, so the definitions agree in this case.

For the Lipschitz bounds we note the following fact about products of Lipschitz functions.

**Lemma 5.3.** *If $f, g$ are functions supported on $0 < |\lambda| \ll 1$ with $|f(\lambda_2) - f(\lambda_1)| \le C_f |\lambda_2 - \lambda_1|^\gamma$, $|g(\lambda_2) - g(\lambda_1)| \le C_g |\lambda_2 - \lambda_1|^\gamma$. If, for all $0 < |\lambda| \ll 1$, we have $|f(\lambda)| \le M_f$ and $|g(\lambda)| \le M_g$, then*

$$|fg(\lambda_2) - fg(\lambda_1)| \lesssim (C_g M_f + C_f M_g)|\lambda_2 - \lambda_1|^\gamma.$$

The proof follows from (19) and the triangle inequality:

$$|fg(\lambda_2) - fg(\lambda_1)| = |f(\lambda_1)[g(\lambda_2) - g(\lambda_1)] + [f(\lambda_2) - f(\lambda_1)]g(\lambda_2)|.$$

The quantities $M_f, M_g$ may be functions of $\lambda$ that become singular as $\lambda \to 0$. The same argument may be applied to see that

$$|\partial_\lambda(fg)(\lambda_2) - \partial_\lambda(fg)(\lambda_1)| \lesssim (C_g M_{f'} + C_{f'} M_g + C_f M_{g'} + C_{g'} M_f)|\lambda_2 - \lambda_1|^\gamma.$$

When we use this in the expansions below, only one of these bounds gets large near zero, the remaining quantities are bounded. This allows us to push forward Lipschitz bounds while also accounting for any singularities.

To obtain the dispersive bounds, we need a slightly longer expansion for $(M \pm (\lambda) + S_1)^{-1}$.

**Lemma 5.4.** *Assume that zero is not regular, and that $|V(x)| \lesssim \langle x \rangle^{-\delta}$. For fixed $0 \le \ell \le 1$, for sufficiently small $|\lambda|$ if $\delta > 3 + 2\ell$ we have*

$$(M^\pm(\lambda) + S_1)^{-1} = D_1 + \lambda D_1 v \mathcal{G}_1 v^* D_1 + \lambda^2 D_1 \Gamma_2^\pm D_1 + D_1 M_{1,\ell}^\pm(\lambda) D_1,$$

*where $\Gamma_2^\pm$ are $\lambda$ independent, absolutely bounded operators, and $M_{1,\ell}^\pm(\lambda)$ satisfies*

$$\|\partial_\lambda^k M_{1,\ell}^\pm(\lambda)\|_{HS} \lesssim \lambda^{2+\ell-k}, \qquad k = 0, 1.$$

*Furthermore, if $\delta > 3 + 2(\gamma + \ell(1-\gamma))$ for some $0 \le \gamma \le 1$, then for $0 < |\lambda_1| \le |\lambda_2| \ll 1$, we have*

$$\|M_{1,\ell}^\pm(\lambda_2) - M_{1,\ell}^\pm(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^\gamma |\lambda_2|^{1+\gamma+\ell},$$
$$\|\partial_\lambda M_{1,\ell}^\pm(\lambda_2) - \partial_\lambda M_{1,\ell}^\pm(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^\gamma |\lambda_2|^{(1+\ell)(1-\gamma)}.$$

Here we keep the first two terms in the expansion explicit since their exact form will be important when we invert the operators $B^\pm(\lambda)$.

*Proof.* Recall the definition $M^\pm(\lambda)$ in (10), the second expansion for $\mathcal{R}_0^\pm$ in Lemma 3.1, and that $T_0 + S_1$ is invertible on $L^2$ from Definition 3.4. With a slight abuse of notation, we write $D_1 = (T_0 + S_1)^{-1}$. This agrees with our previous notation in the case when $S_1 = 0$. Expanding in a Neumann series, we have

$$\begin{aligned}
(M^\pm(\lambda) + S_1)^{-1} &= \big(T_0 + S_1 + \lambda v \mathcal{G}_1 v^* + i\lambda^2 v \mathcal{G}_2^\pm v^* + v \mathcal{E}_2^\pm(\lambda) v^*\big)^{-1} \\
&= \big(\mathbb{1} + D_1(\lambda v \mathcal{G}_1 v^* + i\lambda^2 v \mathcal{G}_2^\pm v^* + v \mathcal{E}_2^\pm(\lambda) v^*\big)^{-1} D_1 \\
&= D_1 - \lambda D_1 v \mathcal{G}_1 v^* D_1 + \lambda^2 D_1 [v \mathcal{G}_1 v^* D_1 v \mathcal{G}_1 v^* - i v \mathcal{G}_2^\pm v^*] D_1 + D_1 M_{1,\ell}^\pm(\lambda) D_1
\end{aligned}$$

Here one needs $\delta > 3+2\ell$ to ensure that the operator with integral kernel $v(x)\mathcal{E}_2^{\pm}(\lambda, |x-y|)v^*(y)$ is Hilbert-Schmidt. The bounds on the error term follow from the error bounds on $\mathcal{E}_2^{\pm}(\lambda)$ in Lemma 3.1, the subscript $\ell$ indicates the extra powers of $\lambda$ in its upper bounds.

The Lipschitz bounds follow from Lemma 5.3 since we may write $M_{1,\ell}^{\pm}(\lambda)$ as a combination of absolutely bounded operators with powers of $\lambda$ and $v\mathcal{E}_2^{\pm}(\lambda)v^*$. In these combinations, any term with no $v\mathcal{E}_2^{\pm}(\lambda)v^*$ has at least three powers of $\lambda$. The Lipschitz bounds on $\mathcal{E}_2^{\pm}(\lambda)$ and its derivative in Lemma 3.2, along with the fact that Lipschitz bounds apply to functions of the form $f(\lambda) = \lambda^k$ for any $k \in \mathbb{N}$. All terms in $M_{1,\ell}$ are dominated by the contribution of $v\mathcal{E}_2^{\pm}(\lambda)v^*$. □

**Lemma 5.5.** *Assume that zero is not regular, and that* $|V(x)| \lesssim \langle x\rangle^{-\delta}$. *For fixed* $0 \le \ell \le 1$, *for sufficiently small* $0 < |\lambda| \ll 1$ *if* $\delta > 3+2\ell$ *we have*

$$(B^{\pm}(\lambda))^{-1} = -\frac{D_2}{\lambda} + D_2\Gamma_0^{\pm}D_2 + B_{-1,\ell}^{\pm}(\lambda),$$

*where* $\Gamma_0^{\pm}$ *are* $\lambda$ *independent absolutely bounded operators.*

$$\|\partial_\lambda^k B_{-1,\ell}^{\pm}(\lambda)\|_{HS} \lesssim \lambda^{\ell-k}, \qquad k = 0, 1.$$

*Furthermore for fixed* $0 \le \gamma \le 1$ *if* $\delta > 3 + 2(\gamma + \ell(1-\gamma))$, *then for* $0 < |\lambda_1| \le |\lambda_2| \ll 1$, *we have*

$$\|B_{-1,\ell}^{\pm}(\lambda_2) - B_{-1,\ell}^{\pm}(\lambda_1)\|_{HS} \lesssim |\lambda_2-\lambda_1|^{\gamma}|\lambda_1|^{(1-\gamma)(\ell-1)}$$
$$\|\partial_\lambda B_{-1,\ell}^{\pm}(\lambda_2) - \partial_\lambda B_{-1,\ell}^{\pm}(\lambda_1)\|_{HS} \lesssim |\lambda_2-\lambda_1|^{\gamma}|\lambda_1|^{\ell(1-\gamma)-\gamma-1}.$$

*Proof.* Recalling that $S_1D_1 = D_1S_1 = S_1$, to use the inversion technique of Lemma 5.2, we first need to invert the operators

$$\begin{aligned} B^{\pm}(\lambda) &= S_1 - S_1(M^{\pm}(\lambda)+S_1)^{-1}S_1 \\ &= S_1 - S_1\Big[D_1 + \lambda D_1 v\mathcal{G}_1 v^* D_1 + \lambda^2 D_1\Gamma_2^{\pm}D_1 + D_1 M_{1,\ell}^{\pm}(\lambda)D_1\Big]S_1 \\ &= -\lambda S_1 v\mathcal{G}_1 v^* S_1 - \lambda^2 S_1\Gamma_2^{\pm}S_1 + B_{1,\ell}^{\pm}(\lambda) \end{aligned}$$

The leading $S_1$ is canceled out by the leading contribution of the second term. Here $B_{1,\ell}^{\pm}(\lambda)$ obeys the same bounds as $M_{1,\ell}^{\pm}(\lambda)$ since $S_1$ is a $\lambda$ independent $L^2$-bounded projection. Defining $B_{0,\ell}^{\pm}(\lambda) = -\lambda^{-1}B_{1,\ell}^{\pm}(\lambda)$, we see that

$$\|\partial_\lambda^k B_{0,\ell}^{\pm}(\lambda)\|_{HS} \lesssim \lambda^{1+\ell-k}, \qquad k = 0, 1.$$

By Lemma 6.3 below, the operator $S_1v\mathcal{G}_1v^*S_1$ is invertible on $S_1L^2$. We denote $D_2 := (S_1v\mathcal{G}_1v^*S_1)^{-1}$. Then, by a Neumann series expansion we have

$$\begin{aligned} (B^{\pm}(\lambda))^{-1} &= -\frac{1}{\lambda}\left[S_1v\mathcal{G}_1v^*S_1 + \lambda S_1\Gamma_2^{\pm}S_1 + B_{0,\ell}^{\pm}(\lambda)\right]^{-1} \\ &= -\frac{1}{\lambda}\left[\mathbb{1} + \lambda D_2S_1\Gamma_2^{\pm}S_1 + D_2B_{0,\ell}^{\pm}(\lambda)\right]^{-1}D_2 = -\frac{D_2}{\lambda} + D_2\Gamma_0^{\pm}D_2 + B_{-1,\ell}^{\pm}(\lambda), \end{aligned}$$

where we collect all the error terms from the Neumann series into $B^{\pm}_{-1,\ell}$, which by the error bounds in Lemma 5.4 yields

$$\|\partial_\lambda^k B^{\pm}_{-1,\ell}(\lambda)\|_{HS} \lesssim \lambda^{\ell-k}, \qquad k=0,1.$$

For the Lipschitz bounds, we need to consider cases since the power on $\lambda$ may be negative. The error term here is composed of products of $\lambda^{-2}M^{\pm}_{1,\ell}(\lambda)$ and operators of the form $\lambda\Gamma_0^{\pm}$ with $\Gamma_0^{\pm}$ independent of $\lambda$. As such, the limiting factor is the Lipschitz behavior of $\lambda^{-2}v\mathcal{E}_2^{\pm}(\lambda)v^*$, since the remaining terms are dominated by this one. We write $A(\lambda)=\lambda^{-2}\mathcal{E}_2^{\pm}(\lambda)(x,y)$ and $r=|x-y|$ to illustrate where the more stringent decay conditions on $V$ arise.

We consider cases. First, if $|\lambda_2-\lambda_1|\approx|\lambda_2|$, then either $|\lambda_1|\ll|\lambda_2|$ or $\lambda_1$ and $\lambda_2$ have opposite signs with $|\lambda_1|\approx|\lambda_2|$. In either case, by Lemma 3.1 we have

$$|A(\lambda_2)-A(\lambda_1)|\lesssim|\lambda_2|^{\ell}r^{\ell}\approx|\lambda_2-\lambda_1|^{\gamma}|\lambda_j|^{\ell-\gamma}r^{\ell},$$

where $\lambda_j=\lambda_2$ if the exponent is positive and $\lambda_1$ if the exponent is negative. On the other hand, if $|\lambda_2-\lambda_1|\ll|\lambda_2|$ then we must have $|\lambda_1|\approx|\lambda_2|$ where $\lambda_1$ and $\lambda_2$ have the same sign. We may then use the mean value theorem to write

$$|A(\lambda_2)-A(\lambda_1)|=\left|\int_{\lambda_1}^{\lambda_2}\partial_\lambda A(s)\,ds\right|\lesssim|\lambda_2-\lambda_1||\lambda_1|^{\ell-1}r^{\ell};$$

we note that zero is not in the interval over which we integrate. Interpolating that with the bound from the triangle inequality of $|\lambda_2|^{\ell}r^{\ell}$ yields the bound that is dominated by $|\lambda_2-\lambda_1|^{\gamma}|\lambda_1|^{(1-\gamma)(\ell-1)}r^{\ell}$. Here we note that the singular behavior of the derivative can't be improved in the interpolation process since the upper bound from the triangle inequality does not involve powers of $\lambda_1$.

A similar case analysis with the derivative shows the second Lipschitz bound:

$$|\partial_\lambda A(\lambda_2)-\partial_\lambda A(\lambda_1)|\lesssim|\lambda_2-\lambda_1|^{\gamma}|\lambda_1|^{(\ell-1)(1-\gamma)-2\gamma}\langle r\rangle^{\gamma+\ell(1-\gamma)},$$

where the power on $\lambda_1$ simplifies to $\ell(1-\gamma)-\gamma-1$. The assumptions on $\delta$ are need to ensure that the kernel $v(x)\langle x-y\rangle^{\gamma+\ell(1-\gamma)}v^*(y)$ is Hilbert-Schmidt. The contribution of the remaining terms in $B^{\pm}_{-1,\ell}(\lambda)$ are dominated by these bounds. □

The preceding lemmas serve to prove the following expansion of $(M^{\pm})^{-1}(\lambda)$ in the presence of a zero energy eigenvalue.

**Proposition 5.6.** *Assume that zero is not regular, and that $|V(x)|\lesssim\langle x\rangle^{-\delta}$. For fixed $0\leq\ell\leq1$, for sufficiently small $0<|\lambda|$ if $\delta>3+2\ell$ we have*

$$(M^{\pm})^{-1}(\lambda)=-\frac{D_2}{\lambda}+\widetilde{\Gamma}_0^{\pm}+M^{\pm}_{-1,\ell}(\lambda),$$

*where*

$$\|\partial_\lambda^k M^{\pm}_{-1,\ell}(\lambda)\|_{HS}\lesssim\lambda^{\ell-k}, \qquad k=0,1.$$

*Furthermore for fixed* $0 \le \gamma \le 1$ *if* $\delta > 3 + 2(\gamma + \ell(1-\gamma))$, *then for* $0 < |\lambda_1| \le |\lambda_2| \ll 1$, *we have*

$$\|M^{\pm}_{-1,\ell}(\lambda_2) - M^{\pm}_{-1,\ell}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma} |\lambda_1|^{(1-\gamma)(\ell-1)},$$
$$\|\partial_\lambda M^{\pm}_{-1,\ell}(\lambda_2) - \partial_\lambda M^{\pm}_{-1,\ell}(\lambda_1)\|_{HS} \lesssim |\lambda_2 - \lambda_1|^{\gamma} |\lambda_1|^{\ell(1-\gamma)-\gamma-1}.$$

We note that the difference between the '+' and '−' terms is not used in the arguments in Section 4 when zero is regular. Hence the $\pm$ dependence of the order $\lambda^0$ term will not affect our ability to use these bounds in this case.

*Proof.* Using Lemmas 5.2, 5.4, and 5.5, we have

$$\begin{aligned}(M^{\pm})^{-1}(\lambda) &= D_1 + \lambda D_1 v\mathcal{G}_1 v^* D_1 + \lambda^2 D_1 \Gamma^{\pm}_2 D_1 + D_1 M^{\pm}_{1,\ell}(\lambda) D_1 \\ &+ \Big(D_1 + \lambda D_1 v\mathcal{G}_1 v^* D_1 + \lambda^2 D_1 \Gamma^{\pm}_2 D_1 + D_1 M^{\pm}_{1,\ell}(\lambda) D_1\Big) S_1 \left(-\frac{D_2}{\lambda} + D_2 \Gamma^{\pm}_0 D_2 + B^{\pm}_{-1,\ell}(\lambda)\right) S_1 \\ &\qquad \Big(D_1 + \lambda D_1 v\mathcal{G}_1 v^* D_1 + \lambda^2 D_1 \Gamma^{\pm}_2 D_1 + D_1 M^{\pm}_{1,\ell}(\lambda) D_1\Big)\end{aligned}$$

Expanding this, the most singular contribution with respect to the spectral parameter is

$$-D_1 \frac{S_1 D_2 S_1}{\lambda} D_1 = -\frac{S_1 D_2 S_1}{\lambda} = -\frac{D_2}{\lambda}.$$

The next largest contribution with respect to the spectral parameter is the $\lambda^0$ term,

$$\widetilde{\Gamma}^{\pm}_0 := D_1 - D_1 S_1 D_2 S_1 D_1 v\mathcal{G}_1 v^* D_1 - D_1 v\mathcal{G}_1 v^* D_1 S_1 D_2 S_1 D_1 + D_1 S_1 D_2 \Gamma^{\pm}_0 D_2 S_1 D_1.$$

The remaining terms form the error $M^{\pm}_{-1,\ell}$. The error and its first derivative are dominated by the contribution of $D_1 S_1 B^{\pm}_{-1,\ell}(\lambda) S_1 D_1$. The first claim on the error term follows from Lemma 5.5. By an application of Lemma 5.3, the Lipschitz bounds on $B^{\pm}_{-1,\ell}$ in Lemma 5.5 and the absolute boundedness of the various operators suffice to show the Lipschitz bounds for $M^{\pm}_{-1,\ell}$. □

It is convenient to define the function

$$\mu_1(\lambda, x, y) = \lambda^{-1} \mu_0(\lambda, x, y) = \lambda^{-1} \chi(\lambda) [\mathcal{R}^+_0 - \mathcal{R}^-_0](\lambda)(x, y).$$

**Lemma 5.7.** *The following bounds hold:*

$$|\mu_1(\lambda, x, y)| \lesssim \min\bigg(|\lambda|, \frac{1}{|x-y|}\bigg), \qquad |\partial_\lambda \mu_1(\lambda, x, y)| \lesssim 1, \qquad \big|\partial^2_\lambda \mu_1(\lambda, x, y)\big| \lesssim \frac{1}{|\lambda|} + |x-y|.$$

*Furthermore, for any* $\gamma \in [0,1]$ *and* $|\lambda_1| \le |\lambda_2|$, *the following Lipschitz bounds hold:*

$$|\mu_1(\lambda_2, x, y) - \mu_1(\lambda_1, x, y)| \lesssim |\lambda_2 - \lambda_1|^{\gamma} \min\left(|\lambda_2|, \frac{1}{|x-y|}\right)^{1-\gamma}$$
$$|\partial_\lambda \mu_1(\lambda_2, x, y) - \partial_\lambda \mu_1(\lambda_1, x, y)| \lesssim |\lambda_2 - \lambda_1|^{\gamma} \left(\frac{1}{|\lambda_1|} + |x-y|\right)^{\gamma}.$$

The first claim follows by dividing (7) by $\lambda$ and the first Lipschitz bounds follows by interpolation. For the Lipschitz bound on the derivative, a case analysis as in the end of the proof of Lemma 5.5 is required.

To establish the first bound we may select $\ell = 0+$ in the expansion of $(M^{\pm})^{-1}$ in Proposition 5.6. The Lipschitz bounds in Proposition 5.6 restrict our choices for $\gamma$ and $\ell$. We must select $\ell$ so that $\ell(1-\gamma)-\gamma > 0$ to ensure integrability near zero. To do so, we select $1 \geq \ell = \frac{\gamma}{1-\gamma}+$, which restricts $\gamma$ to $0 \leq \gamma < 1/2$. Under these conditions, $3+2(\gamma+\ell(1-\gamma)) = 3+4\gamma+$. To obtain an estimate that is integrable at infinity, we may select $\gamma = 0+$ with $\ell = c\gamma$, for some small $c > 0$.

*Proof of Proposition 5.1.* From (20) and Proposition 5.6, we have

$$\begin{aligned}\mathcal{R}_V^{\pm}(\lambda) =& \mathcal{R}_0^{\pm}(\lambda) - \mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda) + \mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda) \\ & -\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda)v^*\left(-\frac{D_2}{\lambda} + \widetilde{\Gamma}_0^{\pm} + M^{\pm}_{-1,\ell}(\lambda)\right)v\mathcal{R}_0^{\pm}(\lambda)V\mathcal{R}_0^{\pm}(\lambda).\end{aligned}$$

By Lemmas 4.2 and 4.5, we only need to consider the last term. Furthermore, by Proposition 5.6 the operators $\widetilde{\Gamma}_0^{\pm} + M^{\pm}_{-1,\ell}(\lambda)$ satify the hypotheses of Lemmas 4.3 and 4.6, so we need only consider the contribution of $D_2$.

Since $D_2$ is independent of the $\lambda$ and $\pm$, using (21) we have

$$\begin{aligned}&\mathcal{R}_0^{+}V\mathcal{R}_0^{+}v^*\frac{D_2}{\lambda}v\mathcal{R}_0^{+}V\mathcal{R}_0^{+} - \mathcal{R}_0^{-}V\mathcal{R}_0^{-}v^*\frac{D_2}{\lambda}v\mathcal{R}_0^{-}V\mathcal{R}_0^{-} \\ &\qquad = \mu_1 V\mathcal{R}_0^{-}v^*D_2v\mathcal{R}_0^{-}V\mathcal{R}_0^{-} + \mathcal{R}_0^{+}V\mu_1v^*D_2v\mathcal{R}_0^{-}V\mathcal{R}_0^{-} \\ &\qquad\qquad + \mathcal{R}_0^{+}V\mathcal{R}_0^{+}v^*D_2v\mu_1V\mathcal{R}_0^{-} + \mathcal{R}_0^{+}V\mathcal{R}_0^{+}v^*D_2v\mathcal{R}_0^{+}V\mu_1.\end{aligned}$$

We choose to move the singular $\frac{1}{\lambda}$ to the difference $\mathcal{R}_0^{+} - \mathcal{R}_0^{-}$ to take advantage of the cancellation and smallness near zero and use the bounds in Lemma 5.7 directly.

We now consider the first bound in the claim. We consider the contribution of the first term in the equation above; the argument for the other three is similar. The proof follows the argument in Lemma 4.3 replacing one resolvent with $\mu_1(\lambda)$. The bounds on $\mu_1$ in Lemma 5.7 allow us to integrate by parts with no boundary terms

$$\begin{aligned}&-\int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)[(\mathcal{R}_0^{+}-\mathcal{R}_0^{-})V\mathcal{R}_0^{-}v^*\frac{D_2}{\lambda}v\mathcal{R}_0^{-}V\mathcal{R}_0^{-}](\lambda)(x,y)\,d\lambda \\ &\qquad = \frac{1}{it}\int_{\mathbb{R}} e^{-it\lambda}\partial_\lambda[\mu_1V\mathcal{R}_0^{-}v^*D_2v\mathcal{R}_0^{-}V\mathcal{R}_0^{-}](\lambda)(x,y)\,d\lambda.\end{aligned}$$

Then for $k_1, k_2, k_3, k_4 \in \{0,1\}$ with $\sum k_j = 1$, the integrand is composed of sums of operators of the form

$$e^{-it\lambda}\partial_\lambda^{k_1}\mu_1V\partial_\lambda^{k_2}\mathcal{R}_0^{-}v^*D_2v\partial_\lambda^{k_3}\mathcal{R}_0^{-}V\partial_\lambda^{k_4}\mathcal{R}_0^{-}.$$

Now observe that $v\partial_\lambda^{k_3}\mathcal{R}_0^- V\partial_\lambda^{k_4}\mathcal{R}_0^-$ is $L^2$ by the boundedness of $D_2$ along with (23) and (24) in the proof of Lemma 4.3. Now observe that by (15), Lemmas 5.7, 3.6 and 3.7, on the support of $\chi(\lambda)$ we have

$$\sup_{x\in\mathbb{R}^3}\Big\|\big(\partial_\lambda^{k_1}\mu_1 V\partial_\lambda^{k_2}\mathcal{R}_0^- v^*\big)(\lambda)(x,\cdot)\Big\|_{L^2}\lesssim 1.$$

Then, by (23) and the absolute boundedness of $D_2$, we have

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\chi(\lambda)\lambda^{-1}[\mathcal{R}_0^+V\mathcal{R}_0^+v^*D_2v\mathcal{R}_0^+V\mathcal{R}_0^+ - \mathcal{R}_0^-V\mathcal{R}_0^-v^*D_2v\mathcal{R}_0^-V\mathcal{R}_0^-](\lambda)(x,y)\,d\lambda\right|\lesssim\frac{1}{|t|}.$$

Here the spatial integrals are controlled as in the proof of Lemma 4.3.

For the weighted bound, we adapt the proof of Lemma 4.6 to account for the effect of $\mu_1$. Since the smallness in $\lambda$ of the resolvents isn't used in the proof of Lemma 4.6, the bound in (29) is valid if we replace one $\mathcal{R}_0$ with $\mu_1$, the only new term that arises is the contribution of terms involving $\partial_\lambda\mu_1$. We note that

$$\begin{aligned}
&\big|[\partial_\lambda\mu_1(\lambda_2)-\partial_\lambda\mu_1(\lambda_1)]V\mathcal{R}_0(\lambda_2)(x,z_2)\big|\\
&\quad\lesssim|\lambda_2-\lambda_1|^\gamma\int_{\mathbb{R}^3}\left(\frac{1}{|\lambda_1|}+|x-z_1|\right)^\gamma\langle z_1\rangle^{-\delta}\left(\frac{1}{|z_1-z_2|^2}+\frac{|\lambda_2|}{|z_1-z_2|}\right)dz_1\\
&\quad\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_1|^{-\gamma}\langle x\rangle^\gamma\int_{\mathbb{R}^3}\langle z_1\rangle^{\gamma-\delta}\left(\frac{1}{|z_1-z_2|^2}+\frac{1}{|z_1-z_2|}\right)dz_1\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_1|^{-\gamma}\langle x\rangle^\gamma.
\end{aligned}$$

Here we need $\delta>2+\gamma$ to apply Lemma 3.6. In particular, using Lemma 3.7 this shows that

$$\|[\partial_\lambda\mu_1(\lambda_2)-\partial\mu_1(\lambda_1)]V\mathcal{R}_0(\lambda_2)(x,\cdot)v^*(\cdot)\|_{L^2}\lesssim|\lambda_2-\lambda_1|^\gamma|\lambda_1|^{-\gamma}\langle x\rangle^\gamma.$$

From here, the proof follows exactly as in the regular case with the above bound used in place of (30). The restrictions on $\gamma$ arise when using the Lipschitz bounds on $\partial_\lambda(M^\pm)^{-1}$ since we need $\ell(1-\gamma)-\gamma-1>-1$ to ensure integrability near zero. □

**Remark 5.8.** *The constraint on $\gamma$ in Theorem 1.1 is an artifact of the proof. It should be possible to prove similar results for $1/2\le\gamma\le 1$ by using longer expansions. That is, writing*

$$\mathcal{R}_0^\pm(\lambda)=\mathcal{G}_0+\lambda\mathcal{G}_1+i\lambda^2\mathcal{G}_2^\pm+\lambda^3\mathcal{G}_3^\pm+\mathcal{E}_3^\pm(\lambda,|x-y|)$$

*where $\mathcal{G}_3^\pm=\pm\frac{|x-y|}{3}\alpha\cdot\hat{e}-\frac{|x-y|}{2}$ would allow for an error term of size $\lambda^3(\lambda|x-y|)^\ell$ for any $0\le\ell\le1$. Using this in expansions for $M^\pm$, $B^\pm$ and considering longer Neumann series expansions would allow for control of the error terms that avoids the bottleneck in the proof of Proposition 5.1. Since our proof allows for a time-integrable bound, we omit this approach for the sake of brevity.*

## 6. Threshold characterization

For completeness, we prove the claims in Definition 3.4, which connect the existence of threshold eigenvalues with the $L^2$ kernel of the operator $T_0$. The characterization of the threshold is similar to that of the massless two dimensional case, [17], with roots in the massive case in [21, 23] and Schrödinger equation [33, 26, 19]. Though the lack of zero energy resonances simplifies many calculations. Recall that $\mathcal{H} = D_0 + V$.

**Lemma 6.1.** *Assume that* $|V(x)| \lesssim \langle x\rangle^{-2-}$. *If* $\phi \in \ker(T_0)$*, then* $\phi = Uv\psi$ *with* $\psi$ *a distributional solution to* $\mathcal{H}\psi = 0$ *and* $\psi \in L^2(\mathbb{R}^3)$*, that is* $\psi$ *is an eigenfunction. Furthermore,* $\psi \in L^p(\mathbb{R}^3)$ *for all* $p \geq 2$.

*Proof.* Take $\phi \in \ker(T_0)$ for $\phi \in L^2$, so

$$0 = T_0\phi = U\phi + v\mathcal{G}_0 v^*\phi = 0 \quad \Longrightarrow \quad \phi = -Uv\mathcal{G}_0 v^*\phi.$$

Define $\psi := -\mathcal{G}_0 v^*\phi$, then $\phi = Uv\psi$. Now, $\mathcal{H} = D_0 + V = -i\alpha \cdot \nabla + V$,

$$\mathcal{H}\psi = (-i\alpha \cdot \nabla + V)\psi = -i\alpha \cdot \nabla\psi + v^*Uv\psi = -i\alpha \cdot \nabla(\mathcal{G}_0 v^*\phi) + v^*\phi.$$

Here, recalling that $\mathcal{G}_0 = -i\alpha \cdot \nabla G_0$ where $G_0(x, y) = (4\pi|x-y|)^{-1} = (-\Delta)^{-1}(x, y)$, we have

$$-i\alpha \cdot \nabla(\mathcal{G}_0 v^*\phi) = -i\alpha \cdot \nabla(-i\alpha \cdot \nabla G_0 v^*\phi) = \Delta(-\Delta)^{-1}v^*\phi = -v^*\phi$$

distributionally. So,

$$\mathcal{H}\psi = -i\alpha \cdot \nabla(\mathcal{G}_0 v^*\phi) + v^*\phi = -v^*\phi + v^*\phi = 0.$$

That is, if $\phi \in \ker(T_0)$ we have $\mathcal{H}\psi = 0$ in the sense of distributions. Now, to show that $\psi \in L^2$, we note that $\psi = -\mathcal{G}_0 v^*\phi$ with $\phi \in L^2$. We can dominate the kernel of $\mathcal{G}_0$ as follows: $|\mathcal{G}_0| \lesssim \mathcal{I}_1$ where $\mathcal{I}_1$ is the fractional integral operator with integral kernel $\mathcal{I}_1(x, y) = c|x-y|^{-2}$. By Lemma 2.3 in [32] $\mathcal{I}_1 \colon L^{2,\sigma} \to L^2$ provided $\sigma > 1$. If we assume $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 2$, then $v^*\phi \in L^{2,1+}$, and we conclude that $\psi \in L^2(\mathbb{R}^3)$.

Further, by the Hardy-Littlewood-Sobolev inequality, $\mathcal{I}_1 \colon L^2(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$, hence we have $\psi \in L^6(\mathbb{R}^3)$. Using that $\psi = -\mathcal{G}_0 v^*\phi$ and $\phi = Uv\psi$, we have $\psi = -\mathcal{G}_0 V\psi$

$$|\psi(x)| \lesssim |\mathcal{G}_0 V\psi(x)| \lesssim \int_{\mathbb{R}^3} \frac{|V(y)\psi(y)|}{|x-y|^2}\, dy \leq \| V(y)|x-y|^{-2}\|_{L_y^{\frac{6}{5}}} \|\psi\|_6 \lesssim 1.$$

The last inequality holds uniformly in $x \in \mathbb{R}^3$ provided $|V(y)| \lesssim \langle y\rangle^{-\delta}$ for some $\delta > 1/2$ by Lemma 3.7, hence $\psi \in L^\infty$. □

This argument also shows that zero energy resonances do not exist. If $\psi \in L^{2,-\frac{1}{2}-}$ solves $\mathcal{H}\psi = 0$, the same argument shows we can bootstrap $\psi \in L^2$, hence $\psi$ is an eigenfunction.

We define $S_1$ to be the orthogonal projection onto the kernel of $T$. By standard arguments, $S_1$ is a finite rank projection, see Definition 3.4 above.

**Lemma 6.2.** *Assume that $|V(x)| \lesssim \langle x\rangle^{-2-}$. If $\mathcal{H}\psi = 0$ with $\psi \in L^2$, then $\phi = Uv\psi \in S_1L^2$, i.e. $T_0\phi = 0$.*

*Proof.* If $0 = \mathcal{H}\psi$, then $i\alpha\cdot\nabla\psi = V\psi = v^*\phi$. To show that $\psi = -\mathcal{G}_0v^*\phi$, noting that $\phi = Uv\psi \in L^2 \subseteq L^1_{loc}$, we have that $v^*\phi \in L^1$. Recalling that $\mathcal{G}_0 = -i\alpha\cdot\nabla G_0$, so that $\Delta(-i\alpha\cdot\nabla\mathcal{G}_0)v^*\phi = -v^*\phi$ in the sense of distributions, we see that

$$-i\alpha\cdot\nabla\big[\psi + \mathcal{G}_0v^*\phi\big] = -i\alpha\cdot\nabla\psi + \Delta G_0v^*\phi = v^*\phi - v^*\phi = 0.$$

This shows that $\psi + \mathcal{G}_0v^*\phi$ is annihilated by the gradient, we must have that $\psi + \mathcal{G}_0v^*\phi = (c_1, c_2, c_3, c_4)^T$ is a constant vector. But, $\psi \in L^2$ and the argument in the Lemma above shows that $\mathcal{G}_0v^*\phi \in L^2$. Hence, $(c_1, c_2, c_3, c_4)^T \in L^2$, which necessitates that $c_j = 0$ for each $j$. Thus, $\psi = -\mathcal{G}_0v^*\phi$ as desired.

Noting that $\phi = Uv\psi = -Uv\mathcal{G}_0v^*\phi$, and recalling that $T_0 = U + v\mathcal{G}_0v^*$, $i\alpha\cdot\nabla\psi = V\psi = v^*\phi$, we see that

$$T_0\phi = U\phi + v\mathcal{G}_0v^*\phi = v\psi + v\mathcal{G}_0V\psi = v\psi + v\mathcal{G}_0v^*\phi = v\psi - v\psi = 0.$$

Hence $\phi \in S_1L^2$ as desired. □

Now, we show that $S_1v\mathcal{G}_1v^*S_1$ is always invertible on $S_1L^2$.

**Lemma 6.3.** *We have the identity*

$$\langle\mathcal{G}_0v^*\phi, \mathcal{G}_0v^*\phi\rangle = -\langle v^*\phi, \mathcal{G}_1v^*\phi\rangle. \tag{36}$$

*Furthermore, the kernel of $S_1v\mathcal{G}_1v^*S_1$ is trivial.*

*Proof.* We first note that $\mathcal{G}_0 = -i\alpha\cdot\nabla G_0$, where $G_0 = (-\Delta)^{-1}$, moving to the Fourier side we see:

$$\langle\mathcal{G}_0v^*\phi, \mathcal{G}_0v^*\phi\rangle = \int_{\mathbb{R}^3} \frac{1}{|\xi|^4}\langle A(\xi)\widehat{v^*\phi}, A(\xi)\widehat{v^*\phi}\rangle_{\mathbb{C}^4}\, d\xi$$

where

$$A(\xi) = \begin{pmatrix} 0 & 0 & \xi_3 & -i\xi_1+\xi_2 \\ 0 & 0 & i\xi_1+\xi_2 & -\xi_3 \\ \xi_3 & -i\xi_1+\xi_2 & 0 & 0 \\ i\xi_1+\xi_2 & -\xi_3 & 0 & 0 \end{pmatrix}.$$

We note that $A(\xi)$ is self-adjoint and $A^*(\xi)A(\xi) = |\xi|^2 I_{4\times4}$. From here, we see that

$$\langle\mathcal{G}_0v^*\phi, \mathcal{G}_0v^*\phi\rangle = \int_{\mathbb{R}^3} \frac{1}{|\xi|^2}\langle\widehat{v^*\phi}, \widehat{v^*\phi}\rangle_{\mathbb{C}^4}\, d\xi.$$

On the other hand, we recall the Schrödinger resolvent $R_0(\lambda^2)$ has Fourier transform $(|\xi|^2 - \lambda^2)^{-1}$. Evaluating the Schrödinger resolvent at $-\lambda^2$ for any $\lambda > 0$ in the resolvent set, then one has

$\mathcal{F}(R_0(-\lambda^2)) = (|\xi|^2+\lambda^2)^{-1}$. Using the expansion that $R_0(-\lambda^2) = G_0 + O(\lambda^{0+})$ as $\lambda \to 0$. Recalling that $\mathcal{G}_1 = G_0 I_{4\times 4}$, we have (again going to the Fourier side)

$$\langle v^*\phi, \mathcal{G}_1 v^*\phi\rangle = \lim_{\lambda\to 0}\langle v^*\phi, R_0(-\lambda^2)I_{4\times 4}v^*\phi\rangle = \lim_{\lambda\to 0}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2+\lambda^2}\langle \widehat{v^*\phi}, \widehat{v^*\phi}\rangle_{\mathbb{C}^4}\, d\xi.$$

Applying the dominated convergence theorem, we bring the limit inside the integral to see

$$\langle v^*\phi, \mathcal{G}_1 v^*\phi\rangle = \int_{\mathbb{R}^3}\frac{1}{|\xi|^2}\langle \widehat{v^*\phi}, \widehat{v^*\phi}\rangle_{\mathbb{C}^4}\, d\xi = \langle \mathcal{G}_0 v^*\phi, \mathcal{G}_0 v^*\phi\rangle,$$

as claimed.

Now, take $\phi \in S_1L^2$ in the kernel of $S_1 v\mathcal{G}_1 v^* S_1$. By Lemmas 6.1 and 6.2 we have $\psi = -\mathcal{G}_0 v^*\phi$ and $\phi = Uv\psi$. Since $S_1 v\mathcal{G}_1 v^* S_1\phi = 0$ we have:

$$0 = \langle \phi, S_1 v\mathcal{G}_1 v^* S_1\phi\rangle = \langle v^*\phi, \mathcal{G}_1 v^*\phi\rangle = \langle \mathcal{G}_0 v^*\phi, \mathcal{G}_0 v^*\phi\rangle = \|\psi\|_{L^2}^2.$$

Hence $\psi = 0$, and $Uv\psi = \phi = 0$. □

This shows that $S_1 v\mathcal{G}_1 v^* S_1$ is invertible on $S_1L^2$ as desired. It follows that

$$P_0 = \mathcal{G}_0 v S_1[S_1 v\mathcal{G}_1 v^* S_1]^{-1} S_1 v^*\mathcal{G}_0 = \mathcal{G}_0 v D_2 v^* \mathcal{G}_0$$

The proof of this is follows the argument of Lemma 7.10 in [21], which proved this in the massive two-dimensional case. We do not use this projection, so we leave the proof to the interested reader.

## 7. High Energy

Finally, we control the high energy portion of the evolution to complete the proofs of Theorems 1.2 and 1.3. Here one cannot use the expansions for $\mathcal{R}_V^{\pm}$ developed for the low energy expansions. Instead, we use the limiting absorption principle, [17]:

$$\sup_{\lambda>0}\|\partial_\lambda^k \mathcal{R}_V^{\pm}(\lambda)\|_{L^{2,\sigma+k}\to L^{2,-\sigma-k}} \lesssim 1, \qquad \sigma > \frac{1}{2}, \qquad k = 0,1,2. \tag{37}$$

This requires only that $|V(x)| \lesssim \langle x\rangle^{-1-}$ and that $V$ has continuous entries. For high energy one has a sharper control on decay of the potential, though it requires continuity of the potential.

Here we cannot use the Lipschitz continuity argument invoked in the low energy regime, but instead proceed via integrating by parts in the Stone's formula, (4). We also selectively iterate the resolvent identity by decomposing $\mathcal{R}_0$ into $\mathcal{R}_L$ and $\mathcal{R}_H$ as in the proof of Theorem 4.7, here with an eye on minimizing the growth in the spectral parameter $\lambda$ rather than to limit the needed decay on $V$.

**Proposition 7.1.** *Let $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 1$ with continuous entries. Then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)(x,y)\, d\lambda\right| \lesssim 1.$$

*If $\delta > 2$, then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)(x,y)\,d\lambda\right| \lesssim \frac{1}{|t|}.$$

*Further, if $\delta > 3$ we have the weighted bound*

$$\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda)(x,y)\,d\lambda\right| \lesssim \frac{\langle x\rangle\langle y\rangle}{|t|^2}.$$

When $\lambda$ is bounded away from zero, we recall (31) and note that By the expansions developed in Lemma 3.1, we have (for $k = 0, 1, 2$)

$$|\partial_\lambda^k \mathcal{R}_L^\pm(\lambda)(x,y)| \lesssim \frac{1}{|\lambda|^k|x-y|^2}, \qquad |\partial_\lambda^k \mathcal{R}_H^\pm(\lambda)(x,y)| \lesssim |\lambda|\,|x-y|^{k-1}. \tag{38}$$

In particular, there is only growth in $\lambda$ when $\mathcal{R}_H^\pm$ appears, while $\mathcal{R}_L^\pm$ is more singular in the spatial variables which necessitates iteration of the Born series. A straight forward computation using Lemma 3.7 shows that for $\sigma > k + 1/2$ we have

$$\|\partial_\lambda^k \mathcal{R}_H^\pm(\lambda)(x,\,\cdot\,)\|_{L^{2,-\sigma}} \lesssim |\lambda|\langle x\rangle^{k-1}. \tag{39}$$

In particular, this bound is uniform when $k = 0, 1$. While $\mathcal{R}_L^\pm$ and its derivatives are not locally $L^2$. Accordingly, we write (omitting the $\pm$ for the moment)

$$\mathcal{R}_V = \mathcal{R}_0 - \mathcal{R}_0 V \mathcal{R}_0 + \mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_0. \tag{40}$$

The first term is controlled by Theorem 2.2 and Corollary 2.3. For the second term we need to utilize the difference between the '+' and '−' resolvents in the Stone's formula, while the third term requires more careful and selective iteration. We note that the factor of $\langle\lambda\rangle^{-3-}$ is needed here since each iteration of $\mathcal{R}_0$ or $\mathcal{R}_H$ contributes a growth of size $\lambda$ in the spectral parameter. To ensure the $\lambda$ integral converges at infinity, we must control a growth of size $|\lambda|^2$ and be integrable at infinity. We prove Proposition 7.1 in a series of lemmas.

**Lemma 7.2.** *Let $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 1$. Then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_0^+ V\mathcal{R}_0^+ - \mathcal{R}_0^- V\mathcal{R}_0^-](\lambda)(x,y)\,d\lambda\right| \lesssim 1.$$

*If $\delta > 2$,*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_0^+ V\mathcal{R}_0^+ - \mathcal{R}_0^- V\mathcal{R}_0^-](\lambda)(x,y)\,d\lambda\right| \lesssim |t|^{-1}.$$

*If $\delta > 3$,*

$$\left|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_0^+ V\mathcal{R}_0^+ - \mathcal{R}_0^- V\mathcal{R}_0^-](\lambda)(x,y)\,d\lambda\right| \lesssim \frac{\langle x\rangle\langle y\rangle}{|t|^2}.$$

*Proof.* By (21) and symmetry, it suffices to bound $[\mathcal{R}_0^+ - \mathcal{R}_0^-]V\mathcal{R}_0^+$. For the first claim we write the resolvents on the left as $\mathcal{R}_0 = \mathcal{R}_L + \mathcal{R}_H$ and consider cases. For the contribution of $\mathcal{R}_L$ on the left, we note that the difference of resolvents satisfies both (7) as well as the bounds for $\mathcal{R}_L$ in (38) by the triangle inequality. As a consequence, we have $|[\mathcal{R}_L^+ - \mathcal{R}_L^-](\lambda)(x,y)| \lesssim \min(|\lambda|^2, |x-y|^{-2})$, which then implies

$$|(\mathcal{R}_L^+ - \mathcal{R}_L^-)(\lambda)(x,z)V(z)\mathcal{R}_0^+(\lambda)(z,y)| \lesssim \langle z\rangle^{-\delta}\bigg(\frac{|\lambda|^{0+}}{|x-z|^{2-}|z-y|} + \frac{|\lambda|^{1+}}{|x-z|^{1-}|z-y|^2}\bigg).$$

On the right side we wrote $\mathcal{R}_0 = \mathcal{R}_L + \mathcal{R}_H$ and used (32). If $\mathcal{R}_H$ is on the left, we do not use any cancellation between '+' and '−' resolvents but note that we may multiply by $|\lambda|\,|x-z|$ as needed to ensure the spatial integrals are bounded uniformly in $x$ and $y$, so

$$|\mathcal{R}_H^{\pm}(\lambda)(x,z)V(z)\mathcal{R}_0^+(\lambda)(z,y)| \lesssim \langle z\rangle^{-\delta}\bigg(\frac{|\lambda|^{1+}}{|x-z|^{1-}|z-y|^2} + \frac{|\lambda|^{0+}}{|x-z|^{2-}|z-y|}\bigg).$$

In any case, by applying Lemma 3.6 with $\delta > 0$, we see that the spatial integrals are bounded uniformly in $x, y$. Hence, we have

$$\sup_{x,y\in\mathbb{R}^3}\bigg|\int_{\mathbb{R}} e^{-it\lambda}\widetilde{\chi}(\lambda)\langle\lambda\rangle^{-3-}[\mathcal{R}_0^+V\mathcal{R}_0^+ - \mathcal{R}_0^-V\mathcal{R}_0^-](\lambda)(x,y)\,d\lambda\bigg| \lesssim \sup_{x,y\in\mathbb{R}^3}\int_{\mathbb{R}}\langle\lambda\rangle^{-2-}\,d\lambda \lesssim 1.$$

We consider the second bound. By (7) and (12), the support of the cut-off and the decay of $\langle\lambda\rangle^{-3-}$, there are no boundary terms when we integrate by parts. We note that differentiation of the cut-off and $\langle\lambda\rangle^{-3-}$ is comparable to division by $\lambda$. By the triangle inequality we need to bound

$$\frac{1}{|t|}\int_{\mathbb{R}}\big|\partial_\lambda\big[\widetilde{\chi}(\lambda)\,\langle\lambda\rangle^{-3-}\,\big(\mathcal{R}_0^+ - \mathcal{R}_0^-\big)V\mathcal{R}_0^-(\lambda)(x,y)\big]\big|d\lambda.$$

Using the bounds in (7) and (12), the above integral is dominated by

$$\frac{1}{|t|}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\widetilde{\chi}(\lambda)\,\langle\lambda\rangle^{-1-}\,\langle z\rangle^{-\delta}\left(\frac{1}{|x-z|^{1-}|z-y|^2} + \frac{1}{|z-y|^2} + \frac{1}{|x-z|^2} + \frac{1}{|x-z|}\right)dz\,d\lambda$$
$$\lesssim \frac{1}{|t|}\int_{\mathbb{R}}\langle\lambda\rangle^{-1-}\,d\lambda \lesssim \frac{1}{|t|},$$

where we require $\delta > 2$ to apply Lemma 3.6. In the case when the derivatives don't act on a resolvent, we interpolate between the two bounds for $\mu(\lambda)$ in (7) to bound the difference of resolvents by $|\lambda|^{1+}|x-z|^{-1+}$ to avoid the logarithmic singularity in the spatial integral.

For the final bound we may integrate by parts a second time without boundary terms. Ignoring when the derivative acts on the first two terms, whose contribution is bounded by $|t|^{-1}$ using the argument above, we use (7) and (12) to control

$$\frac{1}{t^2}\int_{\mathbb{R}}\bigg|\widetilde{\chi}(\lambda)\,\langle\lambda\rangle^{-3-}\,\partial_\lambda^2\big[(\mathcal{R}_0^+ - \mathcal{R}_0^-)V\mathcal{R}_0^-(\lambda)(x,y)\big]\,d\lambda\bigg|$$
$$\lesssim \frac{1}{t^2}\int_{\mathbb{R}}\langle\lambda\rangle^{-1-}\int_{\mathbb{R}^3}\langle z\rangle^{-\delta}\bigg(\frac{\langle x\rangle\langle z\rangle + |x-z|}{|x-z|^2} + 1 + \frac{\langle z\rangle\langle y\rangle}{|z-y|}\bigg)dz\,d\lambda \lesssim \frac{\langle x\rangle\,\langle y\rangle}{t^2},$$

where we used $|x-z| \lesssim \langle x\rangle\langle z\rangle$ and require $\delta > 3$ to apply Lemma 3.6. □

**Lemma 7.3.** *Let $|V(x)| \lesssim \langle x\rangle^{-\delta}$ for some $\delta > 1$ with continuous entries. Then,*

$$\sup_{x,y\in\mathbb{R}^3} \bigg| \int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda) \langle\lambda\rangle^{-3-} \mathcal{R}_0^{\pm} V \mathcal{R}_V^{\pm} V \mathcal{R}_0^{\pm}(\lambda)(x,y)\, d\lambda \bigg| \lesssim 1.$$

*If $\delta > 2$ we have*

$$\sup_{x,y\in\mathbb{R}^3} \bigg| \int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda) \langle\lambda\rangle^{-3-} \mathcal{R}_0^{\pm} V \mathcal{R}_V^{\pm} V \mathcal{R}_0^{\pm}(\lambda)(x,y)\, d\lambda \bigg| \lesssim |t|^{-1}.$$

*If $\delta > 3$ we have*

$$\bigg| \int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda) \langle\lambda\rangle^{-3-} \mathcal{R}_0^{\pm} V \mathcal{R}_V^{\pm} V \mathcal{R}_0^{\pm}(\lambda)(x,y)\, d\lambda \bigg| \lesssim \frac{\langle x\rangle\langle y\rangle}{|t|^2}.$$

*Proof.* In this case we do not utilize the difference between the '+' and '−' resolvents, but do selectively iterate. Accordingly, we suppress the ± notation and write

$$\begin{aligned}(41)\quad \mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_0 &= \mathcal{R}_H V \mathcal{R}_V V \mathcal{R}_H \\ &\quad + \mathcal{R}_L V \mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_H + \mathcal{R}_H V \mathcal{R}_V V \mathcal{R}_0 V \mathcal{R}_L + \mathcal{R}_L V \mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_0 V \mathcal{R}_L.\end{aligned}$$

Using (38) and (12), with $k_j \in \{0,1,2\}$ and $k_1 + k_2 = k$, we see that

$$\begin{aligned}\big|\partial_\lambda^k\big(\mathcal{R}_L(\lambda)(x,z)V(z)\mathcal{R}_0(\lambda)(z,y)\big)\big| &\lesssim \int_{\mathbb{R}^3} \frac{\langle z\rangle^{-\delta}}{|\lambda|^{k_1}|x-z|^2}\bigg(\frac{1}{|z-y|^2} + \frac{|\lambda|}{|z-y|}\bigg)|z-y|^{k_2} \\ &\lesssim \langle\lambda\rangle \int_{\mathbb{R}^3} \frac{\langle z\rangle^{-\delta}}{|x-z|^2|z-y|^2}(1+|z-y|^k)\, dz.\end{aligned}$$

Applying Lemma 3.6, if $k = 0,1$ we bound by $\langle\lambda\rangle(1+|x-y|^{-1})$ provided $\delta > 1$. Applying Lemma 3.6 shows that

$$\sup_{x\in\mathbb{R}^3} \|\partial_\lambda^k\big(\mathcal{R}_L V \mathcal{R}_0(\lambda)(x,\cdot)\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle \qquad k = 0,1, \tag{42}$$

provided $\sigma > k + 1/2$ and $\delta > 2$. When $k = 2$ we see that

$$\|\partial_\lambda^2\big(\mathcal{R}_L V \mathcal{R}_0(\lambda)(x,\,\cdot\,)\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle\langle x\rangle, \tag{43}$$

provided $\sigma > 3/2$ and $\delta > 2$.

Using (41), we may express the integral we need to bound as

$$\int_{\mathbb{R}} e^{-it\lambda} \widetilde{\chi}(\lambda)\, \langle\lambda\rangle^{-3-}\, \Gamma_{1,x}(\lambda) V \mathcal{R}_V(\lambda) V \Gamma_{2,y}(\lambda)(x,y) d\lambda \tag{44}$$

where (39), (42) and (43) show that (for $j = 1,2$)

$$\sup_{x\in\mathbb{R}^3} \|\partial_\lambda^k \Gamma_{j,x}(\lambda)\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle, \qquad k = 0,1, \qquad \|\partial_\lambda^2 \Gamma_{j,x}(\lambda)\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle\langle x\rangle, \tag{45}$$

provided $\sigma > k + 1/2$ and $\delta > 1 + k$ for $k = 0,1,2$. The bounds hold for $\Gamma_{j,y}$ as well, and remain valid for the adjoint operators since $V$ is self-adjoint and $(\mathcal{R}_0^{\pm})^* = \mathcal{R}_0^{\mp}$.

The first claim follows by writing the operators in the integrand in terms of the $L^2$ inner product, (45), and the limiting absorption principle (37). Taking $\sigma = \frac{1}{2}+$ and $\delta > 1$, we have

$$\begin{aligned}
|(44)| &\lesssim \left| \int_{\mathbb{R}} \langle\lambda\rangle^{-3-} \left\langle \Gamma_{1,x}^*(\lambda), V\mathcal{R}_V(\lambda)\Gamma_{2,y}(\lambda)\right\rangle_{L^2} d\lambda \right| \\
&\lesssim \int_{\mathbb{R}} \langle\lambda\rangle^{-3-} \|\Gamma_{1,x}^*(\lambda)\|_{L^{2,-\sigma}} \|V\mathcal{R}_V(\lambda)\Gamma_{2,y}(\lambda)\|_{L^{2,\sigma}} d\lambda \\
&\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}^3} \langle\lambda\rangle^{-3-} \|\Gamma_{1,x}\|_{L^{-\sigma}} \|V\|_{L^{2,-\sigma}\to L^{2,\sigma}} \|\mathcal{R}_V\|_{L^{2,\sigma}\to L^{2,-\sigma}} \|V\|_{L^{2,-\sigma}\to L^{2,\sigma}} \|\Gamma_{2,y}\|_{L^{-\sigma}} d\lambda \\
&\lesssim \int_{\mathbb{R}} \langle\lambda\rangle^{-1-} d\lambda \lesssim 1.
\end{aligned}$$

This bound holds uniformly in $x, y \in \mathbb{R}^3$. For the second claim, we integrate by parts. The bounds in (45) above and the decay of $\langle\lambda\rangle^{-3-}$ ensure there are no boundary terms.

$$|(44)| \lesssim \frac{1}{|t|}\int_{\mathbb{R}}\int_{\mathbb{R}^3} \left| \partial_\lambda^{k_1}\widetilde{\chi}(\lambda)\partial_\lambda^{k_2}\langle\lambda\rangle^{-3-}\, \partial_\lambda^{k_3}\Gamma_{1,x}(\lambda)V\partial_\lambda^{k_4}\mathcal{R}_V(\lambda)V\partial_\lambda^{k_5}\Gamma_{2,x}(\lambda)(x,y)d\lambda \right|$$

where $k_j \in \{0,1\}$ and $\sum k_j = 1$. The operators in the integrand may be controlled as in the first claim using (45) and the limiting absorption principle (37) as follows

$$\begin{aligned}
&\|\partial_\lambda^{k_3}\Gamma_{1,x}\|_{L^{2,-(\frac{1}{2}+k_3)-}} \|V\|_{L^{2,-(\frac{1}{2}+k_4)-}\to L^{2,\frac{1}{2}+k_3+}} \|\partial_\lambda^{k_4}\mathcal{R}_V(\lambda)\|_{L^{2,-(\frac{1}{2}+k_4)-}} \\
&\qquad \|V\|_{L^{2,-(\frac{1}{2}+k_5)-}\to L^{2,\frac{1}{2}+k_4+}} \|\partial_\lambda^{k_5}\Gamma_{2,x}(\lambda)\|_{L^{2,-(\frac{1}{2}+k_5)-}} \lesssim \langle\lambda\rangle^2.
\end{aligned}$$

The decay on $V$ is needed to map between weighted spaces, one needs $\delta > 2$ to ensure multiplication by $V$ maps $L^{2,-\frac{1}{2}-} \to L^{2,\frac{3}{2}+}$. Since only one $k_j$ can be nonzero, this suffices to control the spatial integrals and see that

$$\sup_{x,y\in\mathbb{R}^3} |(44)| \lesssim \frac{1}{|t|}\int_{\mathbb{R}}\int_{\mathbb{R}^3} \langle\lambda\rangle^{-1-} d\lambda \lesssim \frac{1}{|t|}.$$

The final bound follows similarly by integrating by parts and noting that $\sum k_j = 2$. In this case again using (45) and (37) we have

$$\begin{aligned}
&\|\partial_\lambda^{k_3}\Gamma_{1,x}\|_{L^{2,-(\frac{1}{2}+k_3)-}} \|V\|_{L^{2,-(\frac{1}{2}+k_4)-}\to L^{2,\frac{1}{2}+k_3+}} \|\partial_\lambda^{k_4}\mathcal{R}_V(\lambda)\|_{L^{2,-(\frac{1}{2}+k_4)-}} \\
&\qquad \|V\|_{L^{2,-(\frac{1}{2}+k_5)-}\to L^{2,\frac{1}{2}+k_4+}} \|\partial_\lambda^{k_5}\Gamma_{2,x}(\lambda)\|_{L^{2,-(\frac{1}{2}+k_5)-}} \lesssim \langle\lambda\rangle^2\langle x\rangle\langle y\rangle.
\end{aligned}$$

Here, one needs $\delta > 3$ since $\max(|k_j - k_i|) = 2$, the mapping between weighted spaces must map between spaces of the form $L^{2,\sigma-} \to L^{2,\sigma+3}$. We have

$$|(44)| \lesssim \frac{1}{|t|}\int_{\mathbb{R}}\int_{\mathbb{R}^3} \langle\lambda\rangle^{-1-} d\lambda \lesssim \frac{\langle x\rangle\langle y\rangle}{|t|^2}.$$

□

Proposition 7.1 follows expanding $\mathcal{R}_V$ as in (40) and applying Theorem 2.2, Corollary 2.3, Lemmas 7.2 and 7.3 to control each term individually.

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