The Heat Equation in Two (or More) Dimensions

MA 436

Let D be a domain in two or more dimensions and $u(x,t)$ the "temperature" of D, where $x = (x_1, x_2, \ldots, x_n)$ is a point in n dimensional space. The usual physical model for the behavior of u requires that u satisfy

$$\frac{\partial u}{\partial t} - \triangle u = 0$$

in D. An appropriate boundary condition would be $u = h$ on ∂D (that is, the temperature on the boundary is specified) and an appropriate initial condition is $u(x, 0) = f(x)$ for $x \in D$.

Here's a derivation of the heat equation. Think of the temperature u as the "thermal energy density" of D, so higher temperature corresponds to higher energy density. More precisely, suppose that a region B in n dimensional space has a constant temperature u; we'll assume that the total amount of thermal energy in B is given by $E = (c_0 + c_1 u)|B|$, where $|B|$ denotes the n-dimensional volume (e.g., area in 2D) of B and c_0 and c_1 are some constants which depend on the material of which B is made and the system of units we use for volumes and temperatures.

Now let B be ANY region contained in D. If u isn't constant on B then we can chop B up into little pieces, on each of which u is effectively constant, compute the energy contained in each, and add. This leads to

$$E = \int_B (c_0 + c_1 u(x,t))\, dV \tag{1}$$

where the integral is over B in the n spatial variables and dV denotes $dx_1\, dx_2 \cdots dx_n$. Since u changes over time, so does E. We can differentiate both sides of equation (1) to find

$$\frac{dE}{dt} = \int_B c_1 \frac{\partial u}{\partial t}(x,t)\, dV \tag{2}$$

where the "offset" constant c_0 drops out.

Now let's compute $\frac{dE}{dt}$ in a different way. We know that heat flows from hot to cold, or high to low density. The gradient vector $\nabla u(x)$ points in the direction of maximum temperature increase at any point x, so it's reasonable to model heat as flowing in the direction of $-\nabla u(x)$ at any point (the heat energy flows "downhill" in the steepest direction, sort of like a fluid). In fact, we'll assume that the flow of heat energy is given by the vector field

$$\mathbf{F} = -\alpha \nabla u \tag{3}$$

for some constant $\alpha > 0$, where the gradient ∇ is only in the spatial (x) variables (so for any fixed time t, $\mathbf{F}$ is just a vector field in $x_1, \ldots, x_n$ that just happens to change with time t). The rate at which thermal energy is leaving any region B at any given time is given by $\int_{\partial B} \mathbf{F} \cdot \mathbf{n} \, dS$, where $\mathbf{n}$ is an outward pointing unit normal vector on ∂B and dS refers to the "surface measure" on ∂B—just the arc length ds in 2D, or surface area in $3D$. The rate Q at which heat energy is ENTERING B is thus

$$Q = -\int_{\partial B} \mathbf{F} \cdot \mathbf{n} \, dS \tag{4}$$

with $\mathbf{n}$ and dS as above. Now apply the divergence theorem to the right side of equation (4) to obtain

$$Q = -\int_{B} \nabla \cdot \mathbf{F} \, dV \tag{5}$$

with dV as above.

If the thermal energy is conserved then we should have $\frac{dE}{dt} = Q$ (the rate that the energy in B is increasing is the rate the energy flows into B). Equations (2) and (5) yield

$$\int_{B} \left(c_1 \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F} \right) dV = 0 \tag{6}$$

for EVERY subdomain B of D. You should be able to convince yourself that if some (continuous) function ϕ defined on a region D integrates to zero over every subdomain B then ϕ must be identically zero (we've done this argument before). This means that we must have

$$c_1 \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

everywhere in D. If we use $\mathbf{F} = -\alpha \nabla u$ we obtain the heat equation

$$\frac{\partial u}{\partial t} - \kappa \triangle u = 0 \tag{7}$$

where $\kappa = \alpha/c_1$ is called the *diffusivity* and $\triangle u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \cdots \frac{\partial^2 u}{\partial x_n^2}$ is the Laplacian of u. We'll usually take $\kappa = 1$ for simplicity.

Boundary and Initial Conditions

As mentioned above, we can specify the temperature u on the boundary of D (this is the Dirichlet boundary condition) and an initial temperature. All in all we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle u &= 0, \quad x \in D, \;\; t > 0 && (8) \\ u(x,t) &= h(x,t), \quad x \in \partial D, \;\; t > 0 && (9) \\ u(x,0) &= f(x), \quad x \in D, && (10) \end{aligned}$$

It turns out that the heat equation (8) with Dirichlet boundary condition (9) and initial condition (10) has a unique solution, at least if h and f, and the domain D, are nice enough.

An common alternative to the Dirichlet boundary condition is the Neumann boundary condition, in which we specify the rate at which heat energy is entering the domain D at the boundary. The vector field $\mathbf{F}$ dictates the direction of heat energy flow, and $\mathbf{F} \cdot \mathbf{n}$ at any point $x \in \partial D$ and time $t > 0$ gives the rate at which heat energy is leaving D near x; more precisely, $\mathbf{F} \cdot \mathbf{n}$ is the rate (energy per time) that heat energy is leaving D per unit length of ∂D (in 2 dimensions) in the vicinity of x. Thus $-\mathbf{F} \cdot \mathbf{n}$ is the rate at which energy is entering, and this is what we can specify as a boundary condition. We thus require that $-\mathbf{F} \cdot \mathbf{n} = g(x,t)$ for some given function g, which leads to (with $\mathbf{F} = -\alpha \nabla u$)

$$\alpha \frac{\partial u}{\partial \mathbf{n}} = g \tag{11}$$

on ∂D, where $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$. Remember, α is known, and we'll usually take it to be 1 anyway. The case $g \equiv 0$ is called the *insulating* boundary condition—no heat can enter or leave D.

We can thus replace the Dirichlet condition (9) with the Neumann condition (11). Again, the resulting equations will have a unique solution.

Steady-State Solutions

If in the Dirichlet data case the function $h(x,t)$ is independent of time then the solution to the heat equation will stabilize, in the long run, and approach a solution $u(x)$ which no longer depends on time. In this case all time derivatives become zero in equation (8) and we obtain

$$\triangle u = 0, \quad x \in D, \tag{12}$$

$$u(x) = h(x), \quad x \in \partial D \tag{13}$$

which is called *Laplace's Equation*. The same holds true in the Neumann data case, provided $\int_{\partial D} g(x)\, ds = 0$.