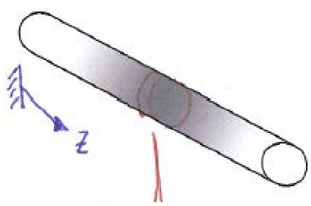
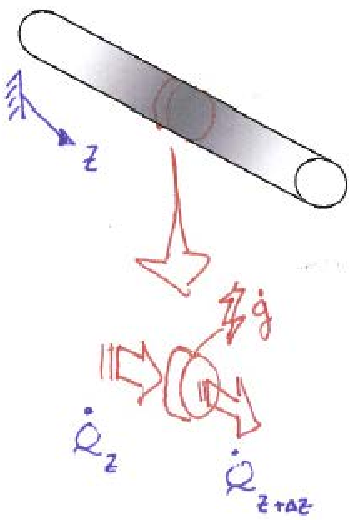
Example

A long cylinder of cross section A is insulated along its outer diameter and is subject to a uniform internal heat generation per unit volume of $\dot{e}_{gen}$. Assuming constant conductivity k and specific heat c, find a differential equation describing the temperature distribution as a function of length and time.



THERMAL ENERGY BALANCE

$$\frac{d}{dt}(U) = \dot{Q}_{IN} - \dot{Q}_{OUT} + \dot{E}_{GEN}$$



THERMAL ENERGY BALANCE

$$\frac{d}{dt}(U) = \dot{Q}_{IN} - \dot{Q}_{OUT} + \dot{E}_{GEN}$$

$$\frac{d}{dt}(\rho A \Delta z\, c T) = -kA\frac{dT}{dz}\Big|_{z} - \left(-kA\frac{dT}{dz}\right)_{z} + \dot{e}_{gen} A(\Delta z)$$

$$\rho c\frac{dT}{dt} = \lim_{\Delta z \to 0}\left(\frac{k\left((dT/dz)_{z+\Delta z} - (dT/dz)_{z}\right)}{\Delta z}\right) + \dot{e}_{gen}$$

$$\boxed{\rho c\frac{dT}{dz} = k\frac{d^2T}{dz^2} + \dot{e}_{gen}}$$

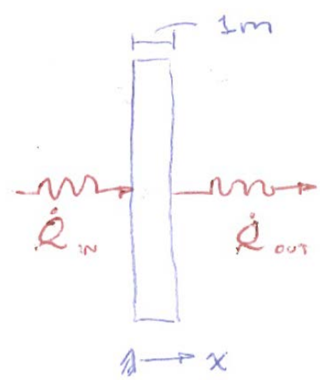
Example

The temperature distribution in a wall 1m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in °C and x is in m. The constants are a = 900°C, b = -300°C/m and c = - 50°C/m². A uniform heat generation $\dot{e}_{gen}$ = 1000 W/m³ exists in the wall. The wall area is 10 m² and has the following properties: ρ = 1600 kg/m³, k = 40 W/m-K and c_p = 4 kJ/kg-K. Determine:

1. the rate of heat transfer entering the wall and leaving the wall. (x=0 and 1 m, respectively),
2. the rate of change of energy storage in the wall, and
3. the time rate of temperature change at x = 0 and 0.25 m.



1) $\dot{Q}_{x=0} = -kA \left.\frac{\partial T}{\partial x}\right|_{x=0} = -kA(b + 2xc)_{x=0}$

$= -40 \frac{W}{m \cdot K} \cdot 10 m^2 \times (-300 \frac{°C}{m} + 2(0)c)$

$= \boxed{120 \text{ kW}}$

$\dot{Q}_{x=1} = -kA(b + 2xc)_{x=1}$

$= -(40)(10)(-300 + (2)(1)(-50)) = \boxed{160 \text{ kW}}$

2) Thermal Energy Balance:

$\frac{dU}{dt} = \dot{Q}_{IN} - \dot{Q}_{OUT} + \dot{E}_{gen} = (120) kW - (160) kW + \dot{e}_{gen} A(1)$

$= 120 - 160 + 1 \frac{kW}{m^3} \times (10 m^2)(1 m)$

$= \boxed{-30 \text{ kW}}$

3) Conduction Eqn:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{gen}}{k}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{gen}}{\rho c}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}(2c) + \frac{\dot{e}_{gen}}{\rho c_p}$$

$$= \frac{40}{(1600)(4)} \times (2)(-80) + \frac{1000}{(1600)(4)}$$

$$= \boxed{-4.69 \times 10^{-4} \ {}^{\circ}C/s}$$

Note this is independent of x.

Also note you can get same result starting from the conduction equation.