Active Learning Exercise – Non-dimensionalization

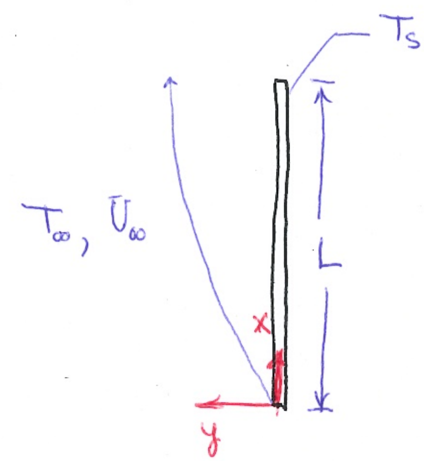
Remember the velocity (momentum) boundary equation (conservation of momentum applied at a point within the boundary layer)?

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

Inertia terms (like $\dot{m}V$ in ConApps)

Viscosity term

Now if we have *buoyancy* as well, we have to add a buoyancy term:



Buoyancy term (NEW!)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}= g\beta\underbrace{(T-T_\infty)}_{\theta} +\nu\frac{\partial^2 u}{\partial y^2}$$

$$= g\beta\theta + \nu\frac{\partial^2 u}{\partial y^2}$$

Non dimensionalization gives us a way to weigh the relative importance of different physical phenomena. One way to arrive at these dimensionless groups is to use the **Buckingham Pi Theorem** to derive the dimensionless groups, or pi terms, directly. Another way is to define dimensionless versions of the variables which show up in the working equations, and then to substitute those variables into the equations. For example, a dimensionless version of the x-direction velocity, u is given by:

$$u^* = u/U_\infty \longrightarrow u = u^* U_\infty$$

Wherever the variable u shows up in the boundary layer equation, then, we would substitute u^*U_∞ instead.

Let us continue with this idea by defining dimensionless versions of the rest of the variables and substituting…

$$v^* = v/U_\infty \longrightarrow v = v^* U_\infty$$

$$x^* = x/L \longrightarrow \frac{\partial}{\partial x} = \left(\frac{\partial x^*}{\partial x}\right)\frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x^*}, \quad \frac{\partial^2}{\partial x^2} = \left(\frac{\partial x^*}{\partial x}\right)^2 \cdot \frac{\partial}{\partial x^{*2}}$$

$$y^* = y/L \longrightarrow \ldots \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y^*}, \quad \frac{\partial^2}{\partial y^2} = \frac{1}{L^2}\frac{\partial^2}{\partial y^{*2}} \qquad = \frac{1}{L^2}\frac{\partial^2}{\partial x^{*2}}$$

$$\theta^* = \frac{\theta}{T_s - T_\infty} \longrightarrow \theta = \theta^*(T_s - T_\infty)$$

$$\therefore \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\theta + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{becomes}$$

$$u^* U_\infty \cdot \frac{1}{L} \cdot \frac{\partial}{\partial x^*}\left[u^* U_\infty\right] + v^* U_\infty \cdot \frac{1}{L}\frac{\partial}{\partial y}\left[u^* U_\infty\right]$$

$$= g\beta\,\theta^*\left[T_s - T_\infty\right] + \nu\frac{1}{L^2}\frac{\partial^2}{\partial y^{*2}}\left[u^* U_\infty\right]$$

$$\frac{U_\infty^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{U_\infty^2}{L}v^*\frac{\partial u^*}{\partial y^*} = g\beta\,\theta^*\left[T_s - T_\infty\right] + \frac{\nu U_\infty}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{g\beta L\left[T_s - T_\infty\right]}{U_\infty^2}\theta^* + \frac{\nu}{U_\infty \cdot L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\nu}{U_\infty \cdot L} \rightarrow \frac{1}{Re_L}$$

$$\rightarrow \frac{L^2\nu^{-2}}{L^2\nu^{-2}} \cdot \frac{g\beta L\left[T_s - T_\infty\right]}{U_\infty^2}$$

$$= \frac{L^2\nu^{-2}}{U_\infty^2} \cdot \frac{g\beta\left[T_s - T_\infty\right]L^3}{\nu^2}$$

$$\frac{1}{Re^2} \cdot Gr$$

& so non-dimensional form of boundary layer momentum is

$$\boxed{u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{Gr}{Re_L^2}\theta^* + \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}}$$

Buoyancy term.

$\frac{Gr}{Re^2} << 1$	$\frac{Gr}{Re^2} \approx 1$	$\frac{Gr}{Re^2} >> 1$
Forced only	Both forced & natural	Natural only