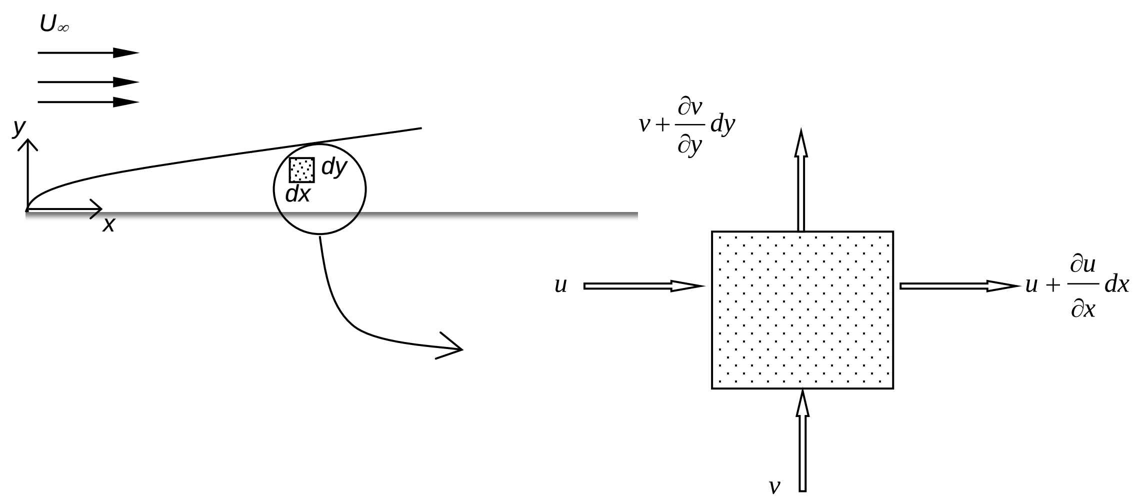
ACTIVE LEARNING EXERCISE: Boundary Layer equations

The boundary layer equations are derived from conservation of mass, momentum, and energy applied to an infinitesimally small point inside a boundary layer. (Remind you of anything? Hmmm?) A few extra assumptions go into their derivation and include:

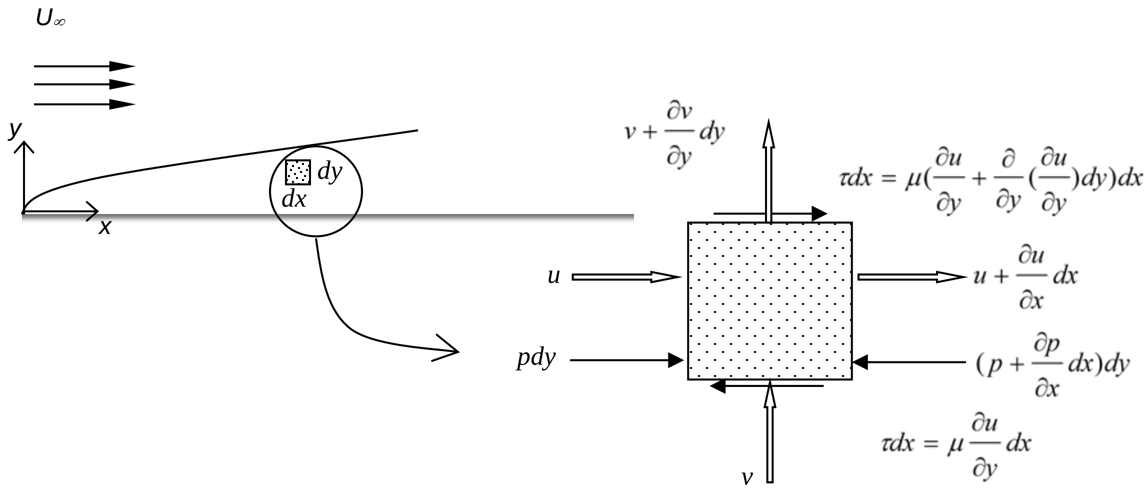
1) The fluid is incompressible and the flow is steady.
2) There are no pressure variations in the direction perpendicular to the plate.
3) Viscosity is constant.
4) Viscous shear stresses in the *y*-direction are negligible.

Below is an illustration of a boundary layer as well as an enlarged view of a fluid element. Conservation of mass applied to the element to (per unit depth into the page) gives



$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,.$$

By drawing all the forces on the fluid element (shown below), you can apply conservation of momentum to derive the second boundary layer equation. Applying the conservation of energy gives the last boundary layer equation.



Here are the results:

Boundary Layer equation from the *x*-momentum conservation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Boundary Layer equation from energy conservation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

Solving the boundary layer equations is a tricky business and involves much mathematical gymnastics. A handy technique known as **scale analysis**, however, lets us estimate the solution to these equations to find out what we *really* want to know (i.e., F_D and h, or C_f and Nu) without solving anything at all! And it's easy too!

Scale Analysis

The fundamental principle of scale analysis is to replace things like u and x in our equations with typical values of those variables. For instance, if we are looking at a plate of length L, where I see x in my equations I would write L That's it! The only difference is that I can't use an equals sign (=) anymore, but an "of the order" thingie (~) instead. This is why I get approximations and not solutions, but it's a lot less work!

Let's start by attempting to estimate the skin friction coefficient:

$$c_f=\frac{\tau_{wall}}{\frac{1}{2}\rho U_\infty^{\ 2}}$$

What is τ equal to?

So now scale it!

This is how we do it. Let’s keep going