Dimensional Analysis, Similitude and Modeling --- Tools for Solving Problems in the Lab

Pre-Lab Activities

(1) Every student should have a lab partner. Do this before you come to your first lab.

(2) Each lab pair should have access to a computer during the lab.

(3) **Read this handout** in conjunction with Section 1.5 of *Fundamentals of Thermal-Fluid Sciences* by Cengel & Turner. Be sure that you *understand*

- the difference between a “dimension” and a “unit,”
- the difference between a “primary” and a “secondary” unit or dimension, and
- what it means to say that an equation is *dimensionally homogeneous*.
- what it means to say that a dimensionally homogeneous equation uses *consistent units.*
- what it means to say than an equation is *dimensionally inconsistent.*

NOTE: For additional help you may consult almost any fluid mechanics text in the library. They have sections on dimensional analysis and/or similitude. For example, Sections 1.4 and 5.1-5.5 in *Fluid Mechanics* by F. White cover this material. Section 1.2 and Chapter 7 of *Fundamentals of Fluid Mechanics* by Munson, Young & Okiishi do as well.

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Introduction

Engineering science education currently emphasizes the development and solution of mathematical models to predict the behavior of engineering systems. Historically, many important engineering problems could not be solved easily or at all using mathematical models. Even today with the availability of high-speed computers and advanced computational techniques, there are still significant problems that cannot for various reasons be solved without recourse to the physical laboratory.

This laboratory will introduce you to a broad class of techniques often called *dimensional analysis.* These techniques allow us to predict the behavior of engineering systems from experiments without having precise mathematical models. The success of this approach, however, does rely on the underlying assumption that mathematical relations do exist between the various physical quantities that describe the system of interest.

Specifically, we will provide you with hands-on experiences that introduce the power and usefulness of three techniques that can be used to model the behavior of physical systems without recourse to a mathematical model:

- Dimensional Analysis (Method of Repeating Variables) ,
- The Method of Parameter Variation, and
- Similitude and Modeling

Lab Objectives—After completing this lab and the associated reading, a student should be able to do the following:

1. Given a mathematical equation that involves physical quantities, determine the dimensions of each term in the equation and whether the equation is dimensionally homogeneous.
2. Given a list of physical variables describing a problem or phenomena, use the *method of repeating variables* to develop a valid set of pi terms.
3. Given a set of pi terms (or developing one as in objective 2) and a set of experimental measurements for the pertinent physical variables, use the *method of parameter variation* to determine the functional relationship between the pi terms.
4. Given a graphical or mathematical relationship between a set of pi terms and the values of the pertinent physical variables, determine the value of an unknown physical variable.
5. Given a scale model of a prototype, use the pertinent pi terms to determine how the important physical variables scale to achieve complete similarity between the prototype and the model.

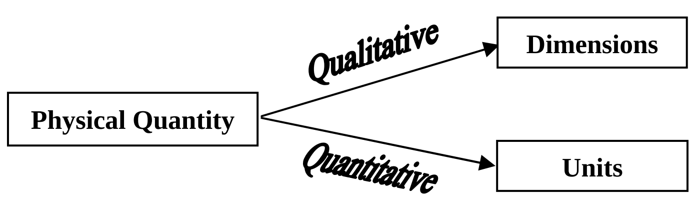
Units vs. Dimensions

Roseyville 127

While driving an interstate in a Great Plains state, we see a sign that says "Roseyville 127." A natural way to interpret the sign is "127 miles to Roseyville." If, however, one mile later there is a ramp for the Roseyville exit, the first sign has been misinterpreted. It was *actually* saying Roseyville had a population of 127. We had no idea what type of thing was being measured.

Let's reconsider the sign. If we arrive at Roseyville in 67 minutes, we are driving at over **110 miles per hour** (which is highly unlikely) *or* the sign presented the distance in **kilometers**.

In the first case, we missed the "qualitative" aspect of the number. This relates to **dimensions**. Was it a length or mass or time, etc.? In the second case, we knew it represented a distance (a length), but missed the quantitative aspect of the number; we had no idea what "127" really meant. What **units** did the number have?



Systems of Dimensions and Units

A dimensional system is a set of **primary (fundamental) dimensions** upon which all other physical quantities can be described. The other physical quantities that can be expressed in terms of the primary dimensions are referred to as **secondary (derived) dimensions**.

Associated with each primary (and secondary) dimension is a corresponding primary (and secondary) unit.

Common dimension systems for mechanics are mass-length-time (*MLT*) or force-length-time (*FLT*).
When thermal effects are important, we must add temperature (θ).
When electrical effects are important, we must add electrical current (I).

The **International System of Units**, the **SI** system, is a *system of units* based on seven primary (base or fundamental) units. The corresponding primary dimensions are time, length, mass, amount of substance, thermodynamic temperature, electric current, and luminous intensity. Each of these units has associated with it a set of operational procedures that can be used to duplicate the seven SI base units in the laboratory.

The most commonly used set of units in the United States is what was once called the **English** system. It is also known as the **United States Customary System** (**USCS**).

MLT AND _FLT_ DIMENSION SYSTEMS

Primary Dimensions & Units

Name	**Symbol**	***MLT* System**				***FLT* System**			
		SI units		**USCS units**		**SI units**		**USCS units**	
Mass	M	kilogram	kg	pound-mass	lbm	---	---	---	---
				slug	slug				
Force	F	---	---	---	---	newton	N	pound-force	lbf
Length	L	meter	m	foot	ft	meter	m	foot	ft
Time	T	second	s	second	s	second	s	second	s

Name	**Symbol**	***MLT* or *FLT* System**			
		SI units		**USCS units**	
Thermodynamic Temperature	θ	kelvin	K	Rankine degree	^{o}R
Current	I	ampere	A	ampere	A
Amount of substance	N	mole	mol	pound-mole	lbmol
Luminous Intensity		candela	cd	candela	cd

Secondary (Derived) Dimensions & Units

Dimensions			**Units**			
Name	***MLT* System**	***FLT* System**	**SI**		**USCS**	
Force	MLT^{-2}	---	newton	N	pound-force	lbf
Mass	---	$FT^{2}L^{-1}$	kilogram	kg	pound-mass	lbm
					slug	slug
Pressure (Stress)	$ML^{-1}T^{-2}$	FL^{-2}	pascal	Pa	---	lbf/in^2
Linear Momentum	MLT^{-1}	FT	---	N·s	---	lbf·s
Electrical Charge	IT	IT	coulomb	C	coulomb	C
Radian	dimensionless	dimensionless	radians	rad**	radians	rad**

**Note that radians by definition are dimensionless as they are the ratio of two lengths; thus when performing unit conversions, it is unnecessary to “cancel out” the radians as they already have units of “1”.

An extended table of dimensions follows:

Dimensions of Common Thermal-Fluid Parameters

Name	Symbol	$MLT\theta$ System	$FLT\theta$ System
Absolute Viscosity	μ	$ML^{-1}T^{-1}$	$FL^{-2}T$
Acceleration	a	LT^{-2}	LT^{-2}
Angle		*1 {dimensionless}*	*1*
Angular Velocity	ω	T^{-1}	T^{-1}
Area	A	L^2	L^2
Density	ρ	ML^{-3}	$FL^{-4}T^{2}$
Dynamic Viscosity	μ	$ML^{-1}T^{-1}$	$FL^{-2}T$
Energy	E	ML^2T^{-2}	FL
Expansion Coefficient	β	θ^{-1}	θ^{-1}
Force	F	MLT^{-2}	F
Frequency	ω	T^{-1}	T^{-1}
Heat Transfer	Q	MLT^{-2}	FL
Kinematic Viscosity	ν	L^2T^{-1}	L^2T^{-1}
Length		L	L
Linear Momentum		MLT^{-1}	FT
Mass	m	M	$FL^{-1}T^{2}$
Mass Flow Rate	$\dot{m}$	MT^{-1}	$FL^{-1}T$
Mass Moment of Inertia		ML^2	FLT^{2}
Modulus of Elasticity		$ML^{-1}T^{-2}$	FL^{-2}
Moment	M	ML^2T^{-2}	FL
Moment of Inertia	I_{xx}	L^4	L^4
Power	$\dot{W}$	ML^2T^{-3}	FLT^{-1}
Pressure	P	$ML^{-1}T^{-2}$	FL^{-2}
Shear Stress	τ	$ML^{-1}T^{-2}$	FL^{-2}
Specific Heat	c, c_p, c_v	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
Specific Weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Surface Tension	σ_s	MT^{-2}	FL^{-1}
Temperature	T	θ	θ
Thermal Conductivity	k	$MLT^{-3}\theta^{-1}$	$FT^{-1}\theta^{-1}$
Time	t	T	T
Torque		ML^2T^{-2}	FL
Velocity		LT^{-1}	LT^{-1}
Volume	V	L^3	L^3
Volumetric Flow Rate	$\dot{V}$, Q	L^3T^{-1}	L^3T^{-1}
Work	W	ML^2T^{-2}	FL

Dimensionally Homogeneous Equations

> "We accept as a fundamental premise that all equations describing physical phenomena must be dimensionally homogeneous."[1]

An equation is *dimensionally homogeneous* if (1) both sides of the equation have identical dimensions and (2) every number that appears in the equation is a pure number.

Occasionally, an equation is incorrectly stated or formulated. Imagine an equation where some terms have units of mass and others have units or energy. Such an equation is "*dimensionally inconsistent*" and being such is invalid as a scientific statement.

In other cases, a number in an equation actually has units, i.e., it is not a pure number. Often such a number is based on variables with specific units and the constant includes, among other things, unit conversions. Both sides of the final equation may have the same dimensions, but the results may be incorrect unless specific units are used. In such a case, the equation may be called "*restricted homogeneous*."

Dimensional Analysis → Finding dimensionless products known as "pi terms"

> "Dimensional analysis is based on the principle that each and every term of a relationship which describes an event in the physical world must have the same dimensions"
>
> -- P. W. Bridgman in *Dimensional Analysis*, Harvard University Press, Cambridge, Massachusetts, 1946.

Buckingham Pi Theorem[2] — Fundamental Theorem of Dimensional Analysis

> If an equation involving *k* variables is dimensionally homogeneous, it can be reduced to a relationship among *k* - *r* independent dimensionless products (pi terms), where *r* is the minimum number of reference dimensions required to describe the variables.

Method of Repeating Variables[3] — One approach to developing the pi terms.

This is one simple, methodical approach to developing the pi terms. This is similar to the method described in a number of fluid mechanics texts. See a *step-by-step procedure* on the following page.

Additional Comments

- How do I know I have the right set of pi terms?

  "Usually our only guideline is to keep the pi terms as simple as possible. Also, it may be that certain pi terms will be easier to work with in actually performing experiments. The final choice remains an arbitrary one and generally will depend on the background and experience of the investigator. It should again be emphasized, however, that although there is no unique set of pit terms for a given problem, the *number* required is fixed in accordance with the pi theorem."[4]

  If you have three pi terms, e.g. Π_1, Π_2, and Π_3, you expect that there is a functional relationship of the form

  $$\Pi_3 = f\,(\,\Pi_2\,,\,\Pi_1\,).$$

  It is equally correct to form a *new* pi term from some combination of the original three as

  $$\Pi_2' = \Pi_2^{\,a}\,\Pi_1^{\,b}$$

  where *a* and *b* are arbitrary exponents. Then the relationship of interest would become

  $$\Pi_3 = f_1\,(\,\Pi_2'\,,\,\Pi_1\,) \quad \text{or} \quad \Pi_3 = f_2\,(\,\Pi_2'\,,\,\Pi_2\,)$$

  even though the number of pi terms has not changed

- Is there a shorter way to find the pi terms? It is possible to form the pi terms by inspection without going through the detailed Method of Repeating Variables. In this approach, the pi terms are formed by merely inspecting the dimensions of the pertinent physical variables. A set of pi terms is correct if it has the correct number of independent and dimensionless pi terms.

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[1] Munson, Young, and Okiishi, *Fundamentals of Fluid Mechanics*, 3rd Ed, J. Wiley, New York, 1990.
[2] *ibid.* Section 7.2.
[3] *ibid.* Section 7.3
[4] *ibid.* Section 7.4.3

Constructing a Set of Dimensionless Parameters

Step 1 List all the important physical variables that are involved in the problem (k = number of variables).

$$y = f(x_1, x_2, x_3, x_4, \ldots, x_{k-1})$$

Step 2A Select a dimension system, usually *FLT* or *MLT*.

Step 2B Express each of the variables in terms of the fundamental (primary) dimensions.

Step 2C Determine the number of ***reference dimensions*** required to completely describe the dimensions of the variables (r = number of reference dimensions).

Usually the number of reference dimensions equals the number of primary dimensions that must be used to write the dimensions of the variables. Infrequently, some of the primary dimensions only occur in unique groupings that are repeated in some of the variables. Keep on the lookout for these groups of dimensions. When they occur, they must be treated as a single reference dimension. For example, if mass, length and time only appeared in the groupings LT^{-1} and M then there would only be 2 reference dimensions even though 3 primary dimensions were involved.

Step 3 Determine the required number of pi terms, $n = k - r$.

Step 4 Select independent ***repeating variables*** equal to the number of reference dimensions, r.

Lessons from experience

- Do select variables whose dimensions are as close to "pure" as possible.
- Try to select one repeating variable from each of the following categories:

  Geometry Material Properties External Effects

  where external effects are "any variable that produces or tends to produce a change in the system."
- Do not select the primary *dependent* variable as a repeating variable.
- Do not select any variable as a repeating variable that has a questionable or minor influence on the problem.
- Be sure that the repeating variables are dimensionally independent.

Step 5 Form a pi term by multiplying one of the non-repeating variables by the product of repeating variables each raised to an exponent that will make the combination dimensionless. (Sometimes this can be done by inspection.)

Step 6 Repeat Step 5 for each of the remaining non-repeating variables.

Step 7 Check each of the resulting pi terms to make sure they are dimensionless.

Step 8 Express the final form as a relationship among the pi terms and think about what it means.

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots \Pi_n)$$

Note that the actual functional relationship between the pi terms can only be obtained from mathematical modeling or experimental data.

Method of Parameter Variation

> "Method of *parameter variation ...* can be defined as the procedure of repeatedly determining the performance of some material, process, or device while systematically varying the parameters that define the object of interest or its conditions of operation."
>
> -- Walter G. Vincenti, *What Engineers Know and How They Know It*, The Johns Hopkins University Press, Baltimore, 1990, pg. 139.

Although this approach can be used with mathematical models to investigate the behavior of a system, it is most commonly used with working scale models to experimentally determine the behavior of a system. In theory, each physical variable is varied over its entire range of values while holding all other variables constant. In practice, this can result in a prohibitively large number of experiments, plus the uncertainty with using measurements from a scale model to predict the behavior of the full-scale device, commonly referred to as the *prototype*.

Dimensional analysis helps reduce the number of data points and physical variables that must be exercised by grouping the important variables into a smaller number of dimensionless groups, the pi terms. Then it is only necessary to exercise each pi term while holding the remaining pi terms constant to investigate the behavior of the system in question.

Modeling & Similitude

To achieve *complete similarity* between a *model* and a *prototype*, the values of the pi terms for the model and the prototype must be equal. When done correctly, this results in *geometric, dynamic,* and *kinematic* similarity between the model and the prototype. For many problems, e.g. modeling the Mississippi River, *incomplete similarity* is used to retain the essential features of the prototype in the model.