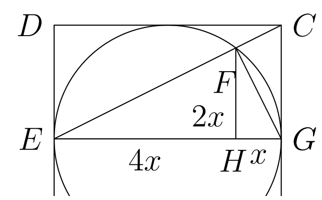
Rose-Hulman High School Math Contest
November 4, 2023

AEBDD DABBA ACEDD BBDEB

11. **A** Draw diameter EG and drop perpendicular FH from F to diameter EG. Triangle EFG is a right triangle that is similar to triangles EHF, FHG, and EGC. Let $GH = x$. Because $EG = 2 \cdot CG$ the length of segment $FH = 2x$ and the length of segment $EH = 4x$. Thus the diameter is $5x = 10$ and $x = 2$. The legs of right triangle EHF are 8 and 4 so $EF = 4\sqrt{5}$.



12. **C** Rewrite in terms of powers of 2 to get $\frac{2^{2x^2}}{2^x} = 2^3$. Take the logarithm base 2 to get the quadratic equation $2x^2 - x = 3$. This has solutions $\frac{1 \pm \sqrt{1+24}}{4} = -1,\ 3/2$.

13. **E** The length of the side of each square is 4. Therefore $AB = \sqrt{12^2 + 4^2} = 4\sqrt{10}$.

Let $BH = x$ then $AH = 3x$, $BJ = 12 - x$, and $JC = 12 - x$. We have $12 - x = JC = GH = 4 + 3x$ so $x = 2$.

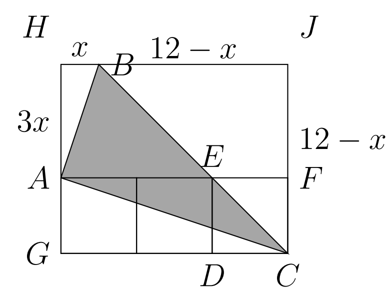
$AB = x\sqrt{10} = 2\sqrt{10}$.

The area of triangle ABC is $\frac{1}{2} \cdot 4\sqrt{10} \cdot 2\sqrt{10} = 40$.

Alternate solution: Angle $ACD = \arctan(1/3)$ and angle $ECF = \arctan(1)$.

Because $90° = \arctan(1) + \arctan(1/2) + \arctan(1/3)$, angle $ACD = \arctan(1/2)$. Thus $AB = 2\sqrt{10}$.

The area of the triangle is $\frac{1}{2} \cdot (4\sqrt{10}) \cdot (2\sqrt{10}) = 40$.



14. **D** The number of subsets is $2^{!0} = 1024$. The number of subsets that have $0, 1$, and 2 elements are $\binom{10}{0} = 1$, $\binom{10}{1} = 10$, and $\binom{10}{2} = 45$, respectively. Therefore the number of subsets with three or more elements is $1024 - 1 - 10 - 45 = 968$.

15. **D** The slope of line ℓ_2 is -2. Therefore $-2 = \frac{3a-14}{a-8}$ which has solution $a = 6$.

16. **B** Every 0 must be followed by a 1 so we are arranging 8 copies of 01 and 2 copies of 1. The number of ways to do this is $\binom{10}{2} = 45$.

17. **B** Expand $(5x-2)^2 = 4x^3 - 16x + 109$ The get $4x^3 - 25x^2 + 4x + 105 = 0$. This is $4(x^3 - \frac{25}{4}x^2 + x + \frac{105}{4}) = 0$. The sum of the roots is the opposite of the coefficient of x^2 in the monic cubic, 25/4

18. **D** There are $12 \cdot 60 = 720$ times displayed on the clock. The times from 1:00 to 6:59 with no repeated digit are 6 choices : 5 choices 8 choices.

The number of times from 1:00 to 6:59 with no repeated digit is $6 \cdot 5 \cdot 8 = 240$.

The times from 7:00 to 9:59 with no repeated digits are 3 choices : 6 choices 8 choices.

The number of times from 7:00 to 9:59 with no repeated digit is $3 \cdot 6 \cdot 8 = 144$.

The times from 10:00 to 10:59 and 12:00 to 12:59 with no repeated digits are 1 2 choices : 4 choices 7 choices.

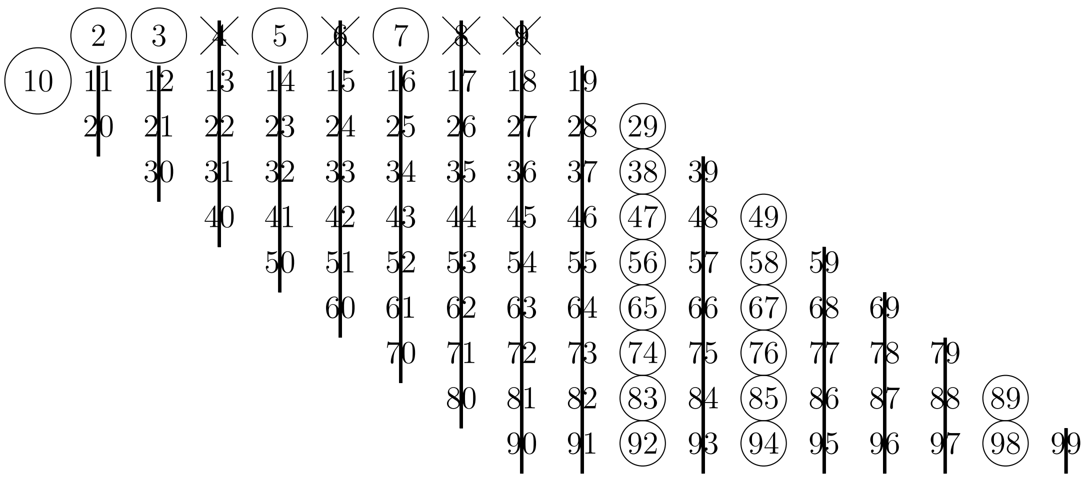
The number of times from 10:00 to 10:59 and 12:00 to 12:59 with no repeated digit is $2 \cdot 4 \cdot 7 = 56$.

The total number of times with no repeated digit is $240 + 144 + 56 = 440$.

Thus the fraction of times with no repeated digit is $\frac{440}{720} = \frac{11}{18} \approx 61\%$.

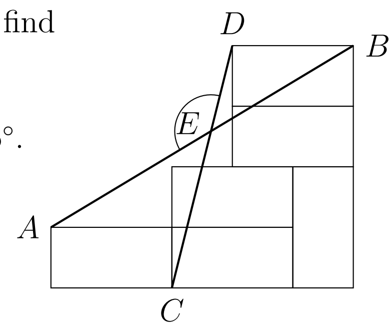
19. **E** Arrange the numbers in columns based on the sum of the digits. All numbers other than 2 in columns listing numbers with an even sum of digits are crossed out.

The single digits 2,3,5, and 7 are circled along with 10, and all numbers with digit sum 11, 13 or 17.



This leaves 21 numbers.

20. **B** The vector $\vec{CD}$ is the vector $\langle 1, 4\rangle$ and the vector $\vec{AB}$ is the vector $\langle 5, 3\rangle$.



We use the dot product formula from analytic geometry to find the angle θ between the vectors: $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos(\theta)$.

$1 \cdot 5 + 4 \cdot 3 = \sqrt{1^2 + 4^2} \cdot \sqrt{5^2 + 3^2} \cos(\theta)$.

Therefore $\cos(\theta) = 1/\sqrt{2}$. Because the angle is obtuse $\theta = 135°$.

Alternate solution The length $DB = 2$. If the lower corner is at $(0, 0)$ then A is at (0,1), B is at (5,4), C is at (2,0), and D is at (3,4).

The equation for the line containing A and B is $y = \frac{3}{5}x + 1$. The equation for the line containing C and D is $y = 4x - 8$. These lines intersect at $E = \left(2 + \frac{11}{17}, \frac{44}{17}\right)$. Therefore the length of ED is $\sqrt{(6/17)^2 + (24/17)^2} = 6/\sqrt{17}$ and the length of EB is $\sqrt{(40/17)^2 + (24/17)^2} = \frac{8\sqrt{2}}{\sqrt{17}}$. Angle DEB is $\alpha = 180° - \theta$.

Apply the law of cosines $(ED)^2 + (EB)^2 - 2(ED) \cdot EB \cdot \cos(\alpha) = (DB)^2$ to get $\frac{36}{17} + \frac{128}{17} - 2\frac{48\sqrt{2}}{17} \cos(\alpha) = 4$ to see that $\cos\alpha = 1/\sqrt{2}$ so $\alpha = 45°$ and DEB is $135°$.