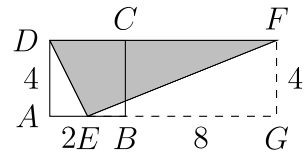
Rose-Hulman High School Math Contest November 12, 2022

1 e	6 d	11 b	16 e
2 a	7 a	12 b	17 b
3 e	8 e	13 c	18 c
4 b	9 a	14 d	19 d
5 c	10 c	15 a	20 a

11. Point E is the midpoint of side AB on square $ABCD$ with $AB = 4$. Right triangle DEF is drawn so that DCF is collinear as shown in the diagram. The area of triangle DEF is ?



Because $\triangle AED$ is similar to $\triangle GFE$ and $GF = 4$, $GE = 8$ so $DE = \sqrt{4^2 + 2^2} = \sqrt{20}$ and $FE = \sqrt{8^2 + 4^2} = 2\sqrt{20}$.

The area of the triangle is $\frac{1}{2} \cdot DE \cdot FE = 20$.

12. The sum of all the real-valued solutions of $|x^2 - 5\,x + 4| = 2$ is ?

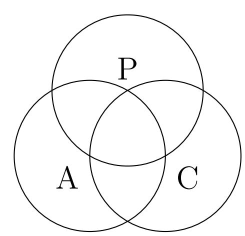
The sum of the solutions to $x^2 - 5x + 4 = 2$ is the coefficient of x^1, so that sum is 5. Because $x^2 - 5x + 4 = (x - 2.5)^2 - 2.25$ the equation $x^2 - 5x + 4 = -2$ has real solutions. The sum of the solutions to $x^2 - 5x + 4 = -2$ is also 5. These are all the solutions to $|x^2 - 5\,x + 4| = 2$ so the sum of all solution is 10.

13. If each interior angle, measured in degrees, of a regular n-gon is $162°$, then n =?

The interior angle of an n-gon is $180\frac{n-2}{n}$ so $180\frac{n-2}{n} = 162$ and $n = 20$.

14. There are 100 orchards in Mason County where apples, cherries, and peaches are grown, and each of these orchards has at least one of these three types of trees. If 77 have apple trees, 57 have cherry trees, 50 have peach trees, and 5 orchards have all three types of these trees, then the number of orchards that have exactly two of the three types of these trees is

Let A represent the set of orchards with apple trees, let C represent the set of orchards with cherry tree, and let P represent the set of orchards with peach trees.



The total number of orchards with exactly two types of trees is $n = |A \cap C| + |A \cap P| + |C \cap P| - 3|A \cap C \cap P|$ so $|A \cap C| + |A \cap P| + |C \cap P| = n + 15$.

The total number of orchards is $100 = |A \cup C \cup P|$, which is

$$\begin{aligned} 100 &= |A| + |C| + |P| - |A \cap C| - |A \cap P| - |C \cap P| + |A \cap C \cap P| \\ &= 77 + 57 + 50 - (n + 15) + 5 \\ &= 174 - n. \end{aligned}$$

Therefore $n = 74$.

15. A right triangle has sides with integer lengths. These three lengths form an arithmetic sequence. One leg of the right triangle has length 2020. The length of the other leg is ?

If the length of the sides are in arithmetic progression we may write the length of the legs as $x-a$ and x, and the length of the hypotenuse as $x+a$. Then $(x+a)^2 = (x-a)^2 + x^2$ so $x^2 + 2ax + a^2 = x^2 - 2ax + a^2 + x^2$. Thus $4ax = x^2$ and because $x \neq 0$, $x = 4a$. So the lengths of the legs are $3a, 4a, 5a$.

Because 2020 is not divisible by 3 we must have $4a = 2020$ so $a = 515$. Thus the length of the other leg is $3 \cdot 515 = 1545$.

16. If n is a positive integer and
$n+(n+1)+(n+2)+\cdots+135+136+137 = 9453$, then $n =$?

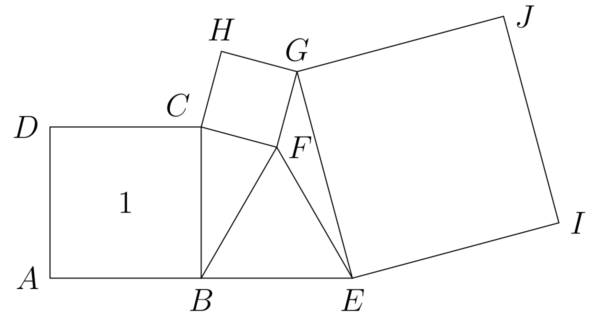
The sum is $(1+2+\cdots 137)-(1+2+\cdots n-1)$ which is $\frac{137}{138}2 - \frac{n(n-1)}{2} = 9453 - n(n-1)/2$.

Set $9453 = 9453 - n(n-1)/2$ to see that $n = 0$ or $n = 1$. Therefore none of the numbers is the solution.

17. Suppose that C is an angle in the first quadrant, with $\sin(C) = c$, and D is an angle in the second quadrant, with $\sin(D) = d$. Then $\sin(C+D) =$?

$\sin(C+D) = \sin(C)\cos(D) + \sin(D)\cos(C)$ Because θ is in the first quadrant $\sin(\theta) = \sqrt{1-\sin^2(\theta)}$. Thus we have $\sin(C+D) = c\sqrt{1-d^2} + d\sqrt{1-c^2}$.

18. Square $ABCD$ with area 1 is adjacent to equilateral triangle BEF. The side lengths of the equilateral triangle are all 1. Square $CFGH$ is constructed above square $ABCD$ and triangle BEF. Square $EIJG$ is constructed to the right of the squares and triangle as shown. The area of square $EIJG$ is ?



If we place B at the origin then vertex E is at $(1,0)$ and vertex F is at $(1/2, \sqrt{3}/2)$.

C is at $(0,1)$ so the change from C to F is $\left(1/2, \frac{\sqrt{3}}{2} - 1\right)$.

The move from F to G is perpendicular to this and is $\left(1 - \frac{\sqrt{3}}{2}, 1/2\right)$ hence vertex G is at $\left(\frac{3}{2} - \frac{\sqrt{3}}{2}, \frac{1}{2} + \frac{\sqrt{3}}{2}\right)$.

Therefore the distance from G to E is $\sqrt{\left(\frac{3}{2} - \frac{\sqrt{3}}{2} - 1\right)^2 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^2}$. Expand the squares to get $\sqrt{\left(\frac{1}{4} - \frac{\sqrt{3}}{2} + \frac{3}{4}\right) + \left(\frac{1}{4} + \frac{\sqrt{3}}{2} + \frac{3}{4}\right)} = \sqrt{2}$.

Because the length of a side is $\sqrt{2}$ the area of the square is 2.

19. A 7-by-7 grid of squares with area 1 is placed in the first quadrant. A circle with center $(0,0)$ is drawn through each lattice point, $(0,1)$, $(1,0)$, $(1,1)$, $(2,0)$, $(0,2)$, $(2,1)$, $(1,2)$, etc. The sum of the areas of the 64 circles is ?

The area of the circle passing through (x, y) is $\pi(x^2+y^2)$. The total area is $\sum_{j=0}^{7}\sum_{k=0}^{7}\pi(j^2+k^2) = \pi\sum_{j=0}^{7}\sum_{k=0}^{7}(j^2+k^2)$.

The sum $\sum_{i=1}^{7} i^2 = \frac{7\cdot8\cdot15}{6} = 140$ so the total area is $\pi\sum_{j=0}^{7}\sum_{k=0}^{7}(j^2+k^2) = \pi\sum_{j=0}^{7} 8j^2 + 140 = \pi(8\cdot140+8\cdot140) = 2240\pi$.

20. A beam of light reflects off the interior sides of a unit square so that the incoming angle the beam makes with the side is the same as the outgoing angle. The beam starts at $(0, 0)$ and first hits a side of the square at $(1, 7/11)$. The beam next hits the side of the square at $(3/7, 1)$ and continues in this way until it hits one of the corners and is stopped. How many times does the beam of light hit the sides of the square before it is stopped?

Instead of reflecting the light beam reflect the squares around a straight beam of light. The beam reaches a corner of the square when the straight beam hits a lattice point. The first lattice point hit is $(7, 11)$. Bouncing off the side of the square corresponds to the straight beam crossing a grid line. The straight beam crosses the grid line $x = 1, 2, \ldots, 10$ and $y = 1, 2, \ldots, 6$ before hitting the lattice point.

Therefore the beam of light bounces off the sides of the square $10 + 6 = 16$ times.