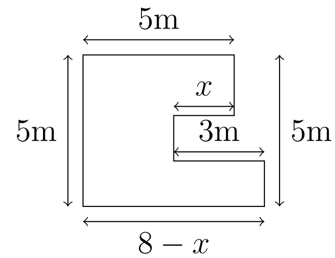
High School Math Contest Prepared by the Mathematics Department of *Rose-Hulman Institute of Technology*, Terre Haute, Indiana. November 16, 2019
Sample solutions

11 If $x = 3$ then the arithmetic sequence $x, x+10, x+20$ consists of three prime numbers, namely, $3, 13, 23$. What is the next smallest value of x so that $x, x+10, x+20$ are all prime numbers?

E None of these. One of x, $x+10$, and $x+20$ is a multiple of 3 so if $x > 3$ then none of the triples consists solely of prime numbers.

12 Determine the perimeter of the polygon.



D 26.

The perimeter is $5+5+5+x+3+8-x = 26$ meters.

13. The number $5/7$ is written in decimal form starting as $0.714\ldots$. What is the sum of the first 2018 digits after the decimal point?

B 9084.

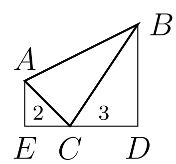
The decimal expansion is $0.\overline{714285}$ with period 6. The sum of the digits in one period is 27.

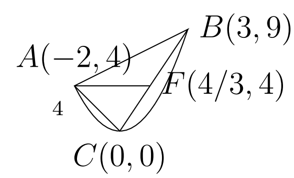
We split $2019 = 6 \cdot 336 + 3$ to see that the first 2019 digits contain 336 periods plus the next three digits 714. Hence the sum of $27 \cdot 336 + 12 = 9084$.

14 The line $y = x + 6$ intersects the parabola $y = x^2$ at points A and B. Point C is the vertex of the parabola $y = x^2$. Compute the area of triangle ABC.

A The line and parabola intersect when $x + 6 = x^2$. This is when $0 = x^2 - x - 6 = (x-3)(x+2)$ so the intersections are at $(-2, 4)$ and $(3, 9)$.

By the shoelace theorem the area is half of $\begin{matrix} 0 & 3 & -2 & 0 \\ 0 & 9 & 4 & 0 \end{matrix} \to (0+12+0)-(0-18+0) = 30$ so the area is 15.





Alternate solution The area is that of trapezoid $ABDE$ minus the areas of triangle ACE and BDC, $5(4+9)/2 - 2(4)/2 - 3(9)/2 = 15$.

Alternate solution: The area is the area of ACF plus the area of BAF, this is $\frac{1}{2}\left[\frac{4}{3} - (-2)\right](4) + \frac{1}{2}\left[\frac{4}{3} - (-2)\right][9-4] = \frac{1}{2}(\frac{10}{3})(9) = 15$.

15 What is the sum of all three digit numbers who digits sum to 9?

D 17,820 To get a sum of 9 we can have $9+0+0$, $k + (9-k) + 0$ for $1 \le k \le 8$, $k + k + (9-2k)$ for $1 \le k \le 4$, or $j + k + (9-j-k)$ with $j < k < (9-j-k)$.

The case $9, 0, 0$ The only number is 900 so the sum of these is 900.

The case $k, 9-k, 0$ For $k = 1$ and $k = 8$ we have $810+180+801+108 = 990+909 = 1899$. Similarly, the sum for other values of k is $990 + 909 = 1899$ so the total for this case is $4 \cdot 1899 = 7596$.

The case $k, k, 9-2k$ For $k = 1$ we have $117 + 171 + 711 = 999$. Similarly, for $k = 2, 4$ the sum is 999. For $k = 3$ the sum is 333. So the total for this case is $3 \cdot 999 + 333 = 3330$.

The case $j < k < 9-j-k$. The only possible distinct triples are 126, 135, and 234. Each of these may be permuted in 6 ways. In the 126 case we have $126+261+612+162+621+216 = 999+999 = 1998$. So the total for this case is $3 \cdot 1998 = 5994$.

The total for all the numbers is $900 + 7596 + 3330 + 5994 = 8496 + 9324 = 17820$.

16 Right triangle ABC has vertices $A(0,0)$, $B(2,0)$, and $C(0,1)$. Right triangle ACD has right angle ACD, $AC = \sqrt{5}$, and $CD = \sqrt{5}/2$. What is the slope of the line containing AD?

E None of these. The triangles are similar so $\angle DAC = \angle CAB$. Let $x = \angle CAB$. Then the slope of the line containing AC is $\frac{1}{2} = \tan(x)$. The slope of the line containing AD is $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} = \frac{1}{3/4} = 4/3$.

Alternatively, the slope of the line containing AC is $1/2$. CD is perpendicular to AC s the slope of the line containing CD is -2. Because $CD = \sqrt{5}/2$ the coordinates of points C and D satisfy $D = C + (-1/2, +1)$. Because C is the point $(2,1)$, D is the point $(3/2, 2)$ so the sloipe of the line containing AD is $\frac{2}{3/2} = 4/3$.

17 The *digital sum* of a number is found by taking the sum of the digits. If the sum is at least ten then take the sum of these digits. Continue this process until the sum is less than 10. For example, $2+0+1+9 = 12$ and $1+2 = 3$ so the digital sum process applied to 2019 ends at 3. Let p be the smallest prime larger than 2019 for which $p+2$ is also prime. Where does the digital sum applied to $p(p+2)$ end?

C 8. We begin by noting the if a number N is of the form $N = 9q + r$ the sum of the digits is of the form $9s + r$ and the digital sum of N is r.

If p is prime then it is not divisible by 3 and so is of the form $3k+1$ or $3k+2$.

If p is of the form $3k+1$ then $p+2 = 3k+3$ is divisible by 3 and is not prime.

Therefore p is of the form $3k+2$.

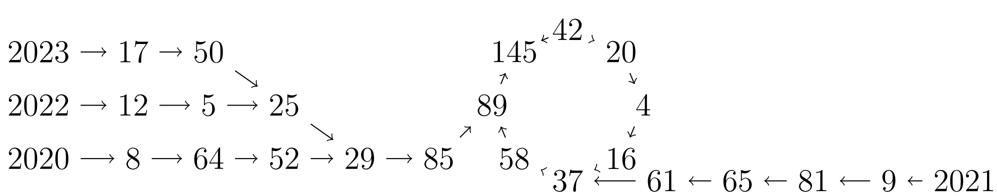
The expression $p(p+2) = p^2 + 2p + 1 - 1 = (p+1)^2 - 1$. Because p is of the form $3k+2$, $p(p+2) = (3k+3)^2 - 1 = 9(k+1)^2 - 1$ so the digital sum is 8.

Alternatively, search to see that the next twin primes after 2019 are 2027 and 2029. Their product is 4,112,783 and the resulting digital sums are 26 and 8.

18 Let the sum of the squares of the digits of a positive integer s_0 be represented by s_1. In a similar way, let the sum of the squares of the digits of s_1 be represented by s_2, and so on. If $s_i = 1$ for some $i \geq 1$, then the original integer s_0 is said to be *happy*. For example, 2019 is happy because $2^2 + 0^2 + 1^2 + 9^2 = 86$, $8^2 + 6^2 = 100$, and $1^2 + 0^2 + 0^2 = 1$. What is the next year after 2019 that is a happy number?

E None of these.

Repeatedly applying the sum of squares of the digits gives



Hence the answer is none of these numbers. The next happy number is 2026.

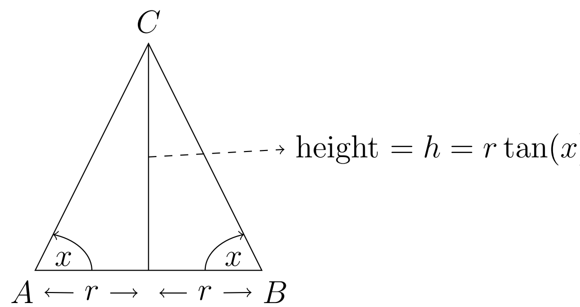
19 Isosceles triangle ABC has $AC = BC$ and the area of triangle ABC is the same as the area of the semi-circle with diameter AB. Let x be the measure of angle BAC. Compute $\tan(x)$ to the nearest one-hundredth.

B 1.57. If the radius of the circle is 1 then the area of the semi-circle is $\pi/2$. The radius of half the base is 1 so the height is $\pi/2$. This means that the slope is $\frac{\pi/2}{1} = \pi/2 \approx 1.57$.

Alternate solution:

$$\frac{1}{2}h(2r) = \frac{1}{2}\pi r^2$$
$$2hr = \pi r^2$$
$$2h = \pi r$$
$$h = \frac{\pi r}{2}$$

Then $\tan(x) = \frac{h}{r} = \frac{\pi r/2}{r} = \frac{\pi}{2} \approx 1.57$.



20 A spinner has the numbers 1,2,3,4 which are formed by wedges of angles $x°$, $2x°$, $3x°$, and $4x°$, respectively. The probability that any of the numbers is spun is proportional to the angle of the corresponding wedge. The spinner is spun four times. What is the probability that the sum of the numbers is a perfect square?

A 181/2000

The probability that n is spun is $n/10$.

The possible squares reached in four spins are 4,9, and 16.

The sum of 4 may be attained by spinning 1,1,1,1. This has probability $(0.1)^4 = 0.0001$.

The sum of 16 may be attained by spinning 4,4,4,4. This has probability $(0.4)^4 = 0.0256$.

The sum of 9 may be attained by spinning 4,3,1,1 in any order; 4,2,2,1 in any order; 3,3,2,1 in any order; or 3,2,2,2 in any order.

There are $\frac{4!}{2!} = 12$ way to spin 4,3,1,1. Each has probability $0.4 \cdot 0.3 \cdot 0.1^2 = 0.0012$. The probability of spinning 4,3,1,1 in some order is 0.0144.

There are $\frac{4!}{2!} = 12$ ways to spin 4,2,2,1. Each has probability $0.4 \cdot 0.2^2 \cdot 0.1 = 0.0016$. The probability of spinning 4,2,2,1 in some order is 0.0192.

There are $\frac{4!}{2!} = 12$ ways to spin 3,3,2,1. Each has probability $0.3^2 \cdot 0.2 \cdot 0.1 = 0.0018$. The probability of spinning 3,3,2,1 in any order is 0.0216.

There are $\frac{4!}{3!} = 4$ ways to spin 3,2,2,2. Each has probability $0.3 \cdot 0.2^3 = 0.0024$. The probability of spinning 3,2,2,2 in any order is 0.0096.

The probability of spinning a total of 9 is $0.0144 + 0.0192 + 0.0216 + 0.0096 = 0.0648$.

The probability of spinning a perfect square is $0.0001 + 0.0256 + 0.0648 = 0.0905 = 181/2000$