11. If $i^2 = -1$ then compute the value of $(1+i)^{50}$.

A. $2^{25}i$

$(1+i)^2 = 1 + 2i + (-1)^2 = 2i$ so $(1+i)^{50} = (2i)^{25} = 2^{25} \cdot i^{24+1}$.
Because $i^4 = 1$ the product is $2^{25}i$

12. What is the sum of the reciprocals of the divisors of 672?

D. 3

Every divisor d of 672 corresponds to a divisor $672/d$. So the sum of the divisors of 672 is $S = \sum_{d|672} d = \sum_{d|672} \frac{672}{d}$. This latter sum is $\sum_{d|672} \frac{672}{d} = 672 \sum_{d|672} \frac{1}{d}$. Thus $S = 672 \sum_{d|672} \frac{1}{d}$ so the sum of the reciprocals of the divisors is $S/672$.

Because $672 = 2^5 \cdot 3 \cdot 7$, the sum of the divisors is $S = \frac{2^6-1}{2-1} \cdot 3^2 - 13 - 1 \cdot 7^2 - 17 - 1 = 63 \cdot 4 \cdot 8$ and $S/672 = \frac{63 \cdot 32}{32 \cdot 21} = 3$.

13. You have three six-sided dice in your pocket. Two are fair dice and one has 6 dots on every side. You take one out of your pocket at random and roll it. Given that you roll a 6, what is the probability that this is a fair die?

B. 1/4

The probability that you take the six-dot die and roll a 6 is $\frac{1}{3} \cdot 1 = \frac{1}{3}$.

The probability that you take a fair die and roll a six is $\frac{2}{3} \cdot \frac{1}{6} = \frac{1}{7}9$.

Therefore the probability that a six is rolled is $\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$.

The probability that the die is fair is $\frac{1/9}{4/9} = \frac{1}{4}$.

14. In a survey there are three possible responses, A,B, and C. Rounded to the nearest percentage 33% of the people responded A, 13% of the people responded B and 53% of the people responded C. What is the smallest possible number of people who responded to the survey?

E. None of these

To get the 13% response rate we note that $\frac{1}{8} < 0.13 < \frac{1}{7}$ so there must be at least 8 people.

If there are 8 people then $0.50 = \frac{4}{8} < .525 < .35 < \frac{5}{8}$ so it is not possible to get a 53% response rate.

For a fraction to fall between $\frac{1}{8}$ and $\frac{1}{7}$ it must be of the form $\frac{r}{s} = \frac{a+b}{8a+7b}$. So the next smallest possible number of people is 15.

We observe that $\frac{2}{15} = .13\overline{3}$, $\frac{5}{15} = .33\overline{3}$, and $\frac{8}{15} = .53\overline{3}$ so it is possible to get these response rates with 15 people.

15 Given a sequence of numbers $a_1, a_2, a_3, \ldots$ such that the next one is the sum of the previous two, i.e. $a_n = a_{n-1} + a_{n-2}$ when $n > 2$. Note that the sequence depends on a_1 and a_2. If $a_7 = 11$ then compute the value of $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$.

D 121.

We use the recursive relationship $a_n = a_{n-1} + a_{n-2}$ to simplify the expression.

Expression	Substitution
$a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}$	
$2a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}$	$a_3=a_1+a_2$
$-a_4+3a_5+a_6+a_7+a_8+a_9+a_{10}$	$2a_5=2a_3+2a_4$
$4a_5+a_7+a_8+a_9+a_{10}$	$-a_4=-a_6+a_5$
$-4a_6+5a_7+a_8+a_9+a_{10}$	$4a_5=4a_7-4a_6$
$-4a_6+5a_7+2a_8+2a_9$	$a_{10}=a_9+a_8$
$-4a_6+7a_7+4a_8$	$2a_9=2a_8+2a_7$
$11a_7$	$4a_8=4a_7+4a_6$

Therefore the sum is $11a_7 = 121$.

16. What is the sum of the digits of 2^{30}?

C 37

The calculation may be somewhat simplified by taking advantage of the fact that $2^{10} = 1024$ and using the binomial theorem.

$2^{30} = (2^{10})^3 = (1000+24)^3 = 10^9 + 3\cdot 24\cdot 10^6 + 3\cdot 24^3\cdot 10^3 + 24^3 = 10^9 + 72\cdot 10^6 + 1728\cdot 10^3 + 13824 = 1,073,741,824$.

The sum of the digits is 37.

17. Write $(2+\sqrt{3})^{2017}$ as $A + B\sqrt{3}$. where A and B are integers. What is the remainder when A is divided by 5?

C 2

Write $(2+\sqrt{3})^n = a_n + b_n\sqrt{3}$. Then $a_0 = 1$, $b_0 = 0$, $a_1 = 2$, $b_1 = 1$. We see that $(2+\sqrt{3})^2 = 7+4\sqrt{3}$ and $(2+\sqrt{3})(a_n+b_n\sqrt{3}) = (2a_n+3b_n) + (a_n+2b_n)\sqrt{3}$.

We build a table of the first few values of a_n, b_n

n	0	1	2	3	4	5	6
a_n	1	2	7	26	97	362	1351
b_n	0	1	4	15	56	209	780

Observe that $(2+\sqrt{3})^3(a_n+b_n\sqrt{3}) = (26a_n+45b_n) + (15a_n+26b_n)\sqrt{3}$. Therefore $a_{n+3} = 26a_n + 45b_n$ and $a_{n-3} - a_n = 5(5a_n+9b_n)$ is a multiple of five. Therefore a_{n+3} and a_n leave the same remainder upon division by five. Thus a_{2017} has the same remainder as a_1. This is 2.

18. What is the number of positive integers less or equal to 900 that have exactly one prime factor in common with 900. For example, 28 is such a number as it shares the prime factor 2 with 900, but they have no other prime factors in common. The number 12 is not such a number because it has multiple prime factors in common with 900; the prime factors 2 and 3.

E None of these

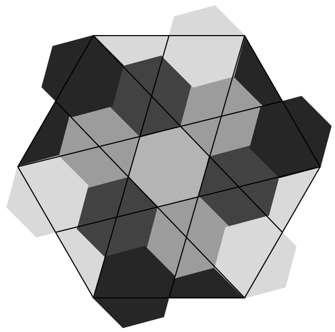
Because $900 = 2^2\cdot 3^2\cdot 5^2$ we are looking for numbers that share exactly one of the factors $2,3,5$ with 900.

We add the positive integers less than 900 that are divisible by 2 or 3 or 5. We must then twice subtract integers that are divisible by two of these $(2,3)$, or $(2,5)$, or $(3,5)$.

The integers divisible by 2, 3, and 5 must be added back three times. So the number is $900\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-2\left(\frac{1}{6}-\frac{1}{10}-\frac{1}{15}\right)+3\cdot\frac{1}{30}\right)=450+300+180-2(150+90+60)+3\cdot 30=930-600+90=420$.

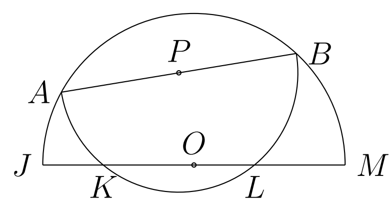
19. The large regular hexagon has an area of 1716. Lines are drawn joining the midpoints of the sides to vertices opposite those sides as shown. Compute the area of the shaded hexagon.

A. 132

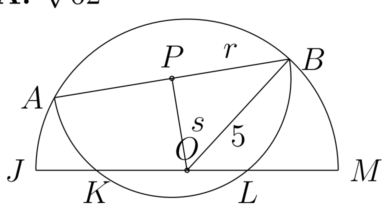


We observe that thirteen copies of the small hexagon exactly fill the large hexagon. Hence the area if the small hexagon is one-thirteenth the area of the large hexagon.

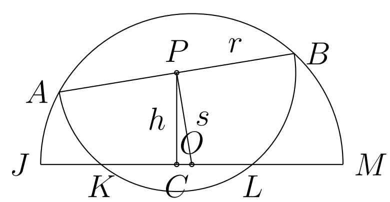
20. $\overline{AB}$ is the diameter of a semi-circle inscribed in the semi-circle of radius 5 centered at O as shown. If $JK = 2$, $KL = 5$, and $LM = 3$ then what is the length AB?



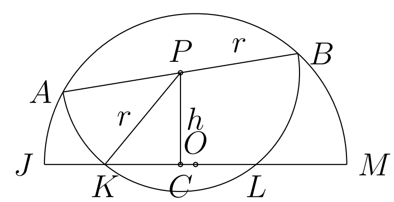
A. $\sqrt{62}$



$s^2 + r^2 = 5^2$



$h^2 + \left(\frac{1}{2}\right)^2 = s^2$



$h^2 + \left(\frac{5}{2}\right)^2 = r^2$

We have the system of equations

$$s^2 + r^2 = 25, \tag{1}$$

$$h^2 + \frac{1}{4} = s^2, \tag{2}$$

$$h^2 + \frac{25}{4} = r^2. \tag{3}$$

Subtract Equation (2) from Equation (3) to get $r^2 - s^2 = 6$. Add this to Equation (1) to see that $2r^2 = 31$.

Therefore $A = 2r = \sqrt{4r^2} = \sqrt{62}$.