2017 High School Math Contest

Rose-Hulman Institute of Technology

Sample solutions

11. A *Niven Number* is a number that is divisible by the sum of its digits. Find the smallest number larger than 2017 that is a Niven Number.

2018 is not divisible by 11. 2019 is not divisible by 12. 2020 is divisible by 4 so the next Niven Number year is 2020. The answer is E. None of these.

12. For how many integers less than 2017 is the smallest odd prime factor either 3 or 5?

The number of these integers divisible by 3 is $\lfloor 2017/3 \rfloor = 672$.

The number of these integers divisible by 5 is $\lfloor 2017/5 \rfloor = 403$.

The integers divisible by 15 are in both these sets and must be counted only once. The number of these integers divisible by 15 is $\lfloor 2017/15 \rfloor = 134$.

Therefore the number of integers is $672 + 403 - 134 = 941$.

13. If the two sentences below are true, which conclusion must also be true?

Either Maggie has a hamster or Bobby has a dog.
Maggie and Bobby are not both dog owners.

A. Bobby has a dog or Maggie has a dog.
B. Maggie has a hamster and Bobby has a dog.
C. Maggie has a hamster or Maggie does not have a dog.
D. Maggie does not have a dog.
E. None of these

Choices A and B are negated by the situation Maggie has a hamster and Bobby has a turtle.

Choice D is negated by Maggie has a dog and Bobby has no pet.

If Maggie has a hamster then C is true. If Maggie does not have a hamster then Bobby has a dog, hence Maggie does not have a dog. Therefore C is again true.

14. How many palindromes with an even number of digits are prime numbers?

In a palindrome with an even number of digits the alternating sum of the digits is zero. For example 2112 has alternating sum $2 - 1 + 1 - 2 = 0$. When the alternating sum of an integer is 0 the integer is divisible by 11. The only prime number divisible by 11 is 11, so there is exactly one palindrome with an even number of digits that is a prime number.

15. A six-sided die has the numbers 1,2,3,4,5,6. Herb rolls the die twice and adds the values. A four sided die has the numbers 1,2,3,4. Zelma rolls this die three times and adds the values. What is the probability that they have the same total?

We compute the probability of each possible sum

Total	2	3	4	5	6	7	8	9	10	11	12
Herb	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
Zelma	0	1/64	3/64	6/64	10/64	12/64	12/64	10/64	6/64	4/64	1/64

For each sum we add the probability that they both roll the same value

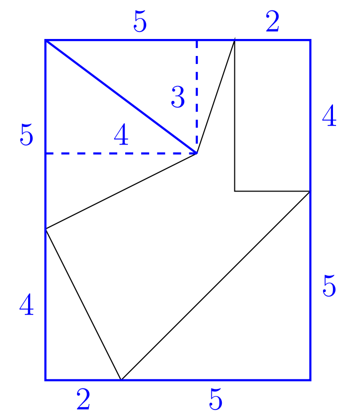
$\frac{1}{36 \cdot 64}(2 \cdot 1 + 3 \cdot 3 + 4 \cdot 6 + 5 \cdot 10 + 6 \cdot 12 + 5 \cdot 12 + 4 \cdot 10 + 3 \cdot 6 + 2 \cdot 4 + 1 \cdot 1) = 47/384.$

16. How many positive integers less than 1000 have at least one digit that is a prime?

The primes are 2,3,5,7. The number of three digit numbers that do not have a prime digit is $5 \cdot 6 \cdot 6 = 180$. The number of two digit numbers that do not have a prime digit is $5 \cdot 6 = 30$. The number of positive one digit integers that do not have a prime digit is 5.

Therefore the number that do have a prime digit is $999 - 180 - 30 - 5 = 784$.

17. The region shown in the diagram is bounded by the lines $y = 4 - 2x$, $y = 4 + x/2$, $y = 3x - 6$, $x = 5$, $y = 5$, and $y = x - 2$, as shown. Determine the area of the region.



Draw the 7-by-9 rectangle enclosing the object and note that the region is created by removing triangles and a rectangle from the 7-by-9 rectangle.

The area is $63 - 4 - 10 - \frac{15}{2} - 8 - \frac{25}{2} = 21$.

18. How many numbers less than 1,000,000,000 (one billion) are divisible by 7 and have the property that the nonzero digits are 5, 1, and 1?

We are looking for combinations of $5 \cdot 10^a + 10^b + 10^c$ that are divisible by seven, where the exponents are integers from $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

Consider the remainders when 10^n is divided by seven

n	0	1	2	3	4	5	6	7	8
10^n	1	3	2	6	4	5	1	3	2
$5 \cdot 10^n$	5	1	3	2	6	4	5	1	3

If the 5 is in either the 10^0 or the 10^6 slot, then the 1s must be in slots that add to 2 or 9. This can be $6 + 3$ (2 ways:n =3 and 1 or), $5 + 4$ (1 way: n=4 and 5), or $1 + 1$ (0 ways: the 5 occupies a slot). Thus there are $2 \times 3 = 6$ such numbers with 5 in the ones or millions slot.

If the 5 is in either the 10^1 or the 10^7 slot, then the 1s must be in slots that add to 6. This can be $5 + 1$ (2 ways), $4 + 2$ (2 ways), or $3 + 3$ (0 ways). Thus there are $2 \times 4 = 8$ such numbers with 5 in the 10^n slot when $n = 1$ or $n = 7$.

If the 5 is in either the 10^2 or the 10^8 slot, then the 1s must be in slots that add to 4 or 11. This could be $6 + 5$ (1 way), $3 + 1$ (4 ways), or $2 + 2$ (0 ways). Thus there are $2 \times 5 = 10$ such numbers with 5 in the 10^n slot when $n = 2$ or $n = 8$.

If the 5 is in the 10^3 slot, then the 1s must be in slots that add to 5 or 12. This could be $6 + 6$ (0 ways), $4 + 1$ (2 ways), or $3 + 2$(4 ways). Thus there are 6 such numbers with 5 in the 10^3 slot.

If the 5 is in the 10^4 slot, then the 1s must be in slots that add to 8. This could be $6 + 2$ (2 ways), $5 + 3$ (2 ways), or $4 + 4$(0 ways). Thus there are 4 such numbers with 5 in the 10^4 slot.

If the 5 is in the 10^5 slot, then the 1s must be in slots that add to 3 or 10. This could be $6+4$ (1 way), $5+5$ (0 ways), $2+1$ (4 ways). Thus there are 5 such numbers with 5 in the 10^3 slot.

Hence, in total, there are $6+8+10+6+4+5=39$ such numbers.

19. Suppose that $f(x)$ is a one-to-one function with $x \in \{1,2,3,4,5,6,7,8,9,10\}$. Suppose further that for each value of x, $f(x) \in \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$. Determine the number of such one-to-one functions for which $f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+f(9)+f(10)$ is a multiple of 3.

The total number of one-to-one functions is 10! multiplied by $\binom{13}{10} = \frac{13\cdot 12\cdot 11}{6} = 286$. The sum of the integers from 1 to 13 is $13(14)/2 = 91$. Thus for the sum of the 10 numbers chosen for the range to add up to a multiple of three, the three numbers omitted must add to one more than a multiple of three. This can happen if the three missing numbers modulo 3 are $\{0,0,1\}$, $\{0,2,2\}$, or $\{1,1,2\}$. The range contains 4 numbers congruent to 0 and 2 modulo 3, and one number congruent to 1 modulo 3.

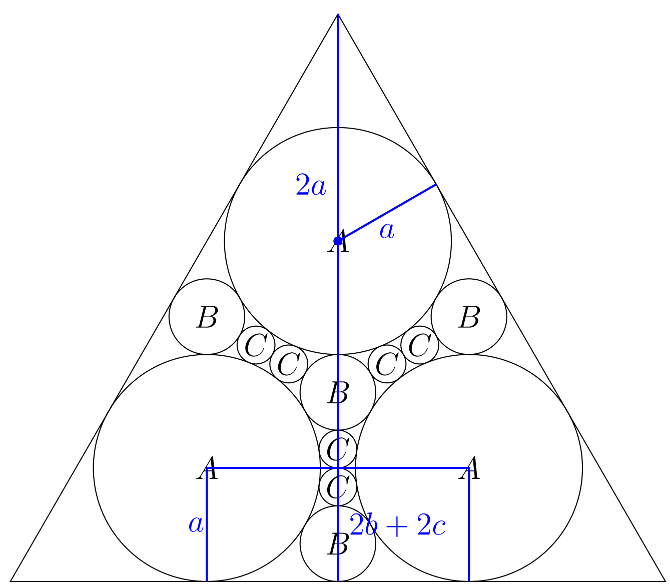
The number of choices for $\{0,0,1\}$ is $\binom{4}{2}\cdot\binom{5}{1} = 30$.

The number of choices for $\{0,2,2\}$ is $\binom{4}{1}\cdot\binom{4}{2} = 24$.

The number of choices for $\{1,1,2\}$ is $\binom{5}{2}\cdot\binom{4}{1} = 40$.

Therefore the number of ways to choose three of the thirteen range elements with a sum congruent to 1 modulo 3 is $30+24+40=94$ ways. The ten elements chosen may be permuted in 10! ways, each creating a different one-to-one function. Therefore the number of such one-to-one functions is $94 \cdot 10!$.

20. Circles are inscribed in an equilateral triangle as shown at the right. The circles labeled A are the same size as each other. The circles labeled B are the same size as each other. The circles labeled C are also the same size as each other. The radius of the circles labeled with A is 30. Determine the radius of a circle labeled C.



Let the radii of circles A, B, and C be a, b, and c, respectively.

By the symmetry of the diagram the center of each circle labeled A is $2a$ units from the nearest vertex. Thus the height of the triangle is $3a+4b+4c$ and the middle circle B has center at the centroid of the equilateral triangle. Thus $3a+b=2(3b+4c)$.

The height of the centers of the largest circles tangent to the base is a. The height is also $2c + 2b$.

Solving this pair of equations, we find that $b = 2c$ and $a = 6c$. As $a = 30$ we have $c = 5$.