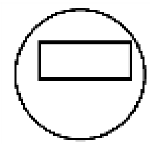
PAGE III solutions

Prob 11. Amy walks at 2 mph. Joe rides his bicycle at 8 mph. If they start together at the same time and travel in opposite directions, in how many minutes will they be 22 miles apart?

a) 2.2 b) 142 c) 122 d) 32 e) none of these

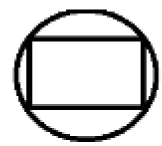
Prob 11. Solution If t is the time in hours then $2t + 8t = 22$ and $t = 2.2$ hrs = 132 min

Prob 12. A rectangle is covered by a circle if every point of the rectangle is also a point of the circle. The figure is an example. The area (in square inches) of the smallest circle that will cover a rectangle with sides 2 inches by 3 inches is



a) 3π b) $13\pi/4$ c) $15\pi/4$ d) 4π e) none of these

Prob 12. Solution The circle must cover the 4 corners and the smallest is shown on the figure. The circle diameter is $\sqrt{2^2 + 3^2}$ and thus the area is $13\pi/4$.



Prob 13. Sally ate 483 grapes in 21 days. Each day she ate 2 more than on the day before. How many did she eat on the fifteenth day?

a) 34 b) 28 c) 29 d) 31 e) none of these

Prob 13. Solution If G is the number eaten on day 1 then
$483 = G + (G+2) + \ldots + (G + 40) = 21G + (2 + 4 + \ldots + 40) = 21G + 420$. Thus $G = 3$. Hence on day 15 she eats $G + 2(14) = 31$.

Prob 14. A box contains 2 red balls and 3 green balls. Jim randomly picks one of the 5 balls and sets it aside. He then randomly picks one of the remaining 4 balls. What is the probability that he has chosen exactly one ball of each color?

a) 3/5 b) 2/5 c) 2/3 d) 1/3 e) none of these

Prob 14. Solution Need first green and second red or first red and second green. Thus
$P = (3/5)(2/4) + (2/5)(3/4) = 3/5$.

Prob 15. How many three digit numbers satisfy the following three conditions: a) the product of the three digits is 72, b) the units digit is equal to or greater than the tens digit and c) the tens digit is equal to or greater than the hundreds digit?

a) 5 b) 6 c) 7 d) 8 e) none of these

Prob 15. Solution List all possible with hundreds digit 1: 11_, 12_, …,19_ and checking the products gives only 189. Continuing with increasing values of the hundreds digit gives the following for all possible: 189, 249, 266, 338 and 346.

PAGE IV solutions

Prob 16. Consider the equation $i\,|\,z\,|^2 + z = 3 + 29i$, where z is complex, $|z|$ is the absolute value of z and $i = \sqrt{-1}$. The sum of the squares of all possible z's satisfying the equation is

a) -25 - 8*i* b) -23 - 6*i* c) -27 - 4*i* d) -22 - 9*i* e) none of these

Prob16. Solution Substituting $z = x + iy$ into the equation gives $i\left(x^2 + y^2\right) + x + iy = 3 + 29i$. Equating real parts gives $x = 3$. Equating imaginary parts gives $9 + y^2 + y = 29$, thus $y = 4$ or $y = -5$. Hence $z = 3 + 4i$ or $z = 3 - 5i$ and $z^2 = 9 + 24i - 16$ or $z^2 = 9 - 30i - 25$. It follows that the sum of the squares of all possible z's satisfying the equation is $-23 - 6i$

Prob 17. The symbol *n*! (*n* factorial) is the product of the integers 1 through *n*. Trailing zeros are those at the end of an integer. For example 165,000 has three trailing zeros. How many trailing zeros does 2015! have?

a) 502 b) 499 c) 582 d) 524 e) none of these

Prob17. Solution For each multiple of 5 in 2015 there will be a corresponding trailing zero. For each multiple of 25 in 2015 there will be two corresponding trailing zero, one has been already counted in our multiples of 5 and one additional trailing zero. Likewise for multiples of 125 and 625. Note that 2015/5 = 403, 2015/25 = 80 + 15/25, 2015/125 = 16 + 15/125 and 2015/625 = 3 + 140/625. Thus the number of trailing zeros in 2015! Is 403 + 80 + 16 + 3 = 502

Prob 18. A number of objects is represented by 43 in base *b* and by 38 in base *B*. If *B** is the smallest value of *B* such that these conditions can be satisfied and *b** the corresponding value of *b*, then the product of *b** and *B** is

a) 48 b) 84 c) 72 d) 56 e) none of these

Prob18. Solution If *N* is the number of objects then $N = 3 + 4b = 8 + 3B$ Since the units digit in base *B* is 8 then *B* > 8. If *B* = 9 then *N* = 35 and *b* =8. Thus (*b**)(*B**) = 72

Prob 19. We call the finite sequence $\{a_0, a_1, a_2, ...a_n\}$, *curious* if a_i is the number of *i*'s in the sequence for each *i* = 0,1,.. . ,*n*. For $n = 3$, the sequences {1,2,1,0} and {2,0,2,0}are examples How many distinct curious sequences are possible with n = 4?

a) 0 b) 2 c) 3 d) 1 e) none of these

Prob 19. Solution Let the sequence be $T = \{a_0, a_1, a_2, a_3, a_4\}$. Since a_i is the number of *i*'s in T then the sum $S = a_0 + a_1 + a_2 + a_3 + a_4 = 5$. Note that $a_0 \neq 0$ since $a_0 = 0$ is self contradictory. We first focus on a_4. If a_4 is 2, 3 or 4 then $S \geq 8$. If $a_4 = 1$, then a_1, a_2 or a_3 must be 4 and, in each of these three cases, *S* will be greater than 5. Thus $a_4 = 0$.

We now focus on a_3. If a_3 is 2, 3 or 4, then $S \geq 6$. If $a_3 = 1$, then $T = \{a_0, a_1, a_2, 1, 0\}$ and a_0, a_1 or a_2 must be 3 but each of these three cases leads to a contradiction. Thus $a_3 = 0$. Since a_4 is also 0 then $a_0 \geq 2$. Since a_1 and a_2 can't be 3 or 4 then $a_0 = 2$. Hence a_1 and a_2 must be 1 or 2. The only pair that does not lead to a contradiction is $a_1 = 1$ and $a_2 = 2$ .Thus only $T = \{2,1,2,0,0\}$ satisfies the required conditions.

Prob 20. Let [*x*] represent the greatest integer which is less than or equal to *x*. For example, [3] = 3 and [2.6] = 2. Let *x* be positive and *k* a positive integer with $1 \leq k \leq 169$. For how many values of *k* does the equation $x[x] = k$ have a solution?

a) 78 b) 74 c) 79 d) 75 e) none of these

Prob 20. Solution If [*x*] > 13 then *k* > 169, so we need consider [*x*] in the range 1 to 13 inclusive. For example if [*x*] = 4, then 4*x* is equal to *k* and thus an integer. Also $4 \leq x < 5$, hence the values of *x* are $[4, 4\frac{1}{4}, 4\frac{1}{2}, 4\frac{3}{4}]$ and the four corresponding values of *k* are [16,17,17,19]. Similar calculations that if [*x*] = 5 then there are 5 corresponding values of *k* namely [25,26,27,28,29]. For [*x*] = 12 the *k* values are $[144,145,...,155]$ and there are 12 possible values of *k*. Similar calculations show that if [*x*] = *i*, then there are *i* possible values of *k*, for $1 \leq i \leq 12$ For [x] = 13, there is only one possible value of *k*, namely 169. Hence the number of possible values of *k* is $1 + \sum_{i=1}^{12} i = 79$.