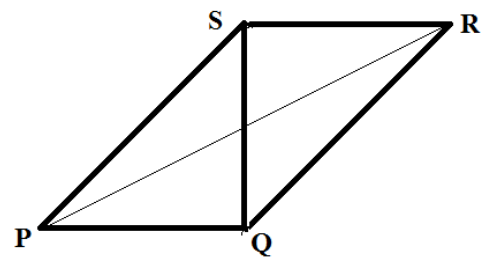
Prob 11. Let $f = i + j^2$, where i and j are non-negative integers. If $0 \le i \le 3$ and $0 \le j \le 3$, then the sum of all possible distinct values of f is

a) 68 b) 72 c) 70 d) 64 e) none of these

11 soln. If $j = 0$, then f = 0,1,2 or 3; if $j = 1$, then f = 1,2,3 or 4; if $j = 2$, then f = 4,5,6 or 7; if $j = 3$, then f = 9,10,11 or 12. Thus f can be any integer from 0 through 12 except 8 and the sum of the possible values of f is

$$\left(\sum_{f=0}^{12} f\right) - 8 = \frac{(12)(13)}{2} - 8 = 70$$

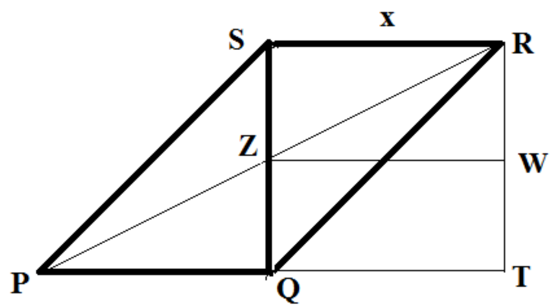
Prob 12. A square is cut along a diagonal and reassembled to form the parallelogram PQRS as shown in the diagram. If PR = 80 inches, what is the area of the original square, in square inches?



a) 1280 b) 1240 c) 1340 c) 1320 c) none of these

12 soln. Let x be the side length of the initial square, Z the intersection of SQ and PR, T the point such that QTRS is a square and W the midpoint of RT. For the right triangle ZWR we have

$$(x/2)^2 + x^2 = 40^2 \quad \Rightarrow \quad x^2 = 1280$$



Prob13. If $f(x) = px + q$ and $f\big(f(f(x))\big) = 8x + 21$, and if p and q are real numbers, then $p^2 + q^2$ is equal to

a) 18 b) 12 c) 16 d) 32 e) none of these

13 soln.

$f(f((x) = p(px + q) = p^2 x + pq$

$f(f(f(x)) = p^3 x + p^2 q + pq = 8x + 21$

$p = 2 \quad q = 3$

Prob 14. The circumferences of two circles are consecutive primes p_1 and p_2, with $p_1 > p_2$. p_1 and p_2 are consecutive primes if there are no primes between p_1 and p_2. Central angles of 30 degrees subtend arcs of lengths s_1 and s_2 on the two circles. If $s_1 + s_2 = 3$ then $p_1 - p_2$ is

a) 1 b) 2 c) 3 d) 4 e) none of these

14 soln. The arc length subtended by a central angle is the radius times the angle in radians. The circle radii are $p_1 / 2\pi$ and $p_2 / 2\pi$. Thus $3 = s_1 + s_2 = (\frac{p_1 + p_2}{2\pi})(\frac{30\pi}{180}) \Rightarrow p_1 + p_2 = 36$. Hence $p_1 = 19$ and $p_2 = 17$.

Prob 15. The difference of two positive numbers is 4 and the product of the two numbers is 19. The sum of the two numbers is

a) $\sqrt{23}$ b) $\sqrt{23}/2$ c $2 + \sqrt{23}$ d) $2\sqrt{23}$ e) none of these

15 soln. Let the numbers be *x* and y, with $x > y$. Then $xy = 19$ and $x - y = 4$. Solving gives $y^2 + 4y - 19 = 0$ and $y = -2 \pm \sqrt{4 + 19}$. Since $y > 0$, then we use the plus branch and the corresponding *x* is $x = 2 + \sqrt{4 + 19}$. Thus $x + y = 2\sqrt{23}$.

Prob 16. A man has walked two-thirds of the distance across a railroad bridge when he sees a train approaching at 54 miles per hour. If *R* is his uniform speed (in miles per hour) such that he can just manage to escape by running to either end of the bridge, then the sum of the digits in *R* is

a) 7 b) 6 c) 9 d) 10 e) none of these

16 soln. Let 2*D* be the distance he has walked across the bridge when he first sees the train, *D* the remaining distance and *R* his required running rate. Let time equal 0 when he first sees the train. If he runs toward the oncoming train then he will just clear the bridge as the train arrives at time *D/R*. If he runs the other direction he will exit the bridge at time *2D/R*. The train will arrive at this end of the bridge at time *D/R* plus the time for the train to cross, that is $D/R+3D/54$. Hence

$$\frac{D}{R}+\frac{3D}{54}=\frac{2D}{R} \quad \Rightarrow \quad R=18.$$

Prob 17, Given a 5 digit number abcde. Put a one in front gives 1abcde. Multiply by 3 gives the 6 digit number abcde1. The sum of the digits in the original 5 digit number abcde is

a) 25 b) 26 c) 27 d) 28 e) none of these

17 soln. (1 a b c d e)(3) = a b c d e 1

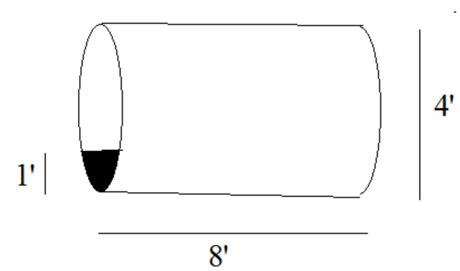
Thus the units digit of (3)(e) is 1 and e = 7. Since (3)(e) will carry 2, then the (3)(d)+2 has units digit 7 and d must be 5. Similar calculations give c = 8, b = 2 and a = 4. Thus abcde = 42857.

Prob 18. Sara and David were reading the same novel. When Sara asked David what page he was reading, he replied that the product of the page number he was reading and the next page number was 100172. The sum of the digits for the page was David reading was

a) 10 b) 11 c) 12 d) 13 e) none of these

18 soln. The page number will be near the square root of 100172. Note that 300^2 = 90000, 310^2 = 96100, 320^2 = 102400. Thus the sqrt(100172) is between 310 and 320 but closer to 320. Also the product of the (consecutive) units digits on the two pages must have units digit 2. Trying the pairs (6,7) and (8,9) for for the units digits, we find that the page numbers are 316 and 317.

Prob 19. A Right circular cylinder with a radius of 2 feet and height of 8 feet is partially filled with oil. When it is placed horizontally on a level surface, as shown in the figure, the top of the oil is 1 foot above this surface. If the cylinder is then placed in a vertical position, with a circular base resting on a level surface, then what is the height (in feet) of the oil above the base?

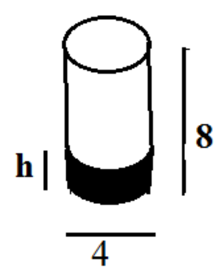


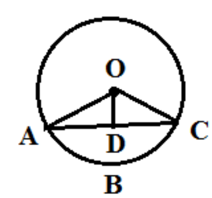
a) $4\pi/3 - 2\sqrt{3}/\pi$ b) $8/3 - 2\sqrt{3}/\pi$ c) $8\pi/3 - 4\sqrt{3}$

d) $8/\pi - \sqrt{3}/3$ e) none of these

19 soln. When the cylinder is horizontal, the volume of oil is

8 [sector ABCD], where [R] denotes the area of region R. When the cylinder is vertical, the volume of oil is $\pi(2^2)h$, where h is the cylinder height.





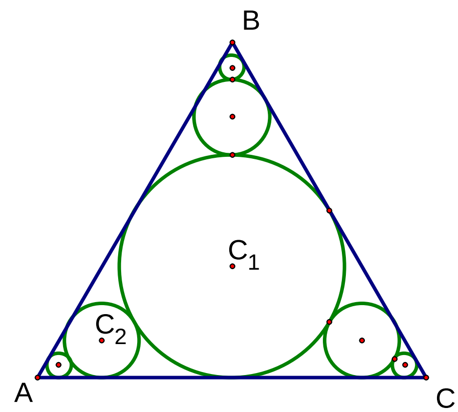
BD =1 , OB = 2 thus OD = 1. Since AO = 2 then $\angle OAD = 30^\circ$ and

$\angle AOC = 120^\circ$. Hence [sector AOCB] = $\pi(2^2)(120/360) = 4\pi/3$.

[segment ABCD] = [sector AOCB] – [triangle AOC] = $4\pi/3 - \sqrt{3}$. Equating the two oil volumes gives

$$8(4\pi/3 - \sqrt{3}) = 4\pi h \text{ and thus } h = 8/3 - 2\sqrt{3}/\pi \cong 1.564$$

Prob 20. Circle C_1 has a radius of 1 and is inscribed in the equilateral triangle *ABC*. The circle C_2 is constructed tangent to C_1, and also tangent to *AB* and *AC*. An infinite sequence of circles is constructed with successive circles tangent to the previous circle and also tangent to sides *AB* and *AC*. Similar sequences of circles are also constructed near the other two vertices. Let *K* be the sum of the areas of all the regions that are inside the triangle but not inside any of the circles. If you use the approximations $\pi \cong 3.142$ and $\sqrt{3} \cong 1.732$, then the resulting approximate value of *K* is nearest to which of the following?



a) 0.872 b) 0.874 c) 0.876 d) 0.878 e) 0.880

20 soln. Construct the right triangle C_1C_2F, where C_1F is perpendicular to *AC* and C_2F is parallel to *AC*. Thus $C_1F = r_1 - r_2$, $C_1C_2 = r_1 + r_2$ and $\angle C_1C_2F = \angle C_1AG = 30^\circ$, since C_1A bisects the 60 degree angle *BAC*. Hence $\sin(30^\circ) = (1 - r_2)/(1 + r_2)$ and $r_2 = 1/3$. Similar calculations show suceeding circles have radii 1/3 that of the previous circle. Thus the area of the sequence of circles near *A* (including C_1) is $\pi(1 + 1/9 + 1/81 + ...) = \pi(\frac{1}{1 - 1/9}) = 9\pi/8$. If we exclude the area of C_1, then the area of the circles C_2 and smaller, near *A*, is $\pi/8$. Hence the sum of the areas of all the circles is $\pi(1 + 3/8)$ and *K* = $3\sqrt{3} - 11\pi/8 \cong 3(1.732) - 11(3.142)/8 = 0.87575$.