Prob. 11. Sally rows 12 miles per hour in still water. In a river with constant flow rate, she rows a certain distance upstream in two hours and rows back the same distance downstream in one hour. At what rate was the river flowing in miles per hour?

a) 1 b) 3 c) 5 d) 7 e) none of these

11. solution. If R be the river rate in mph, then her downstream rate is 12 + R and her upstream rate is 12 – R. If D is the one-way distance in miles, then $1 = D/(12+R)$ and $2 = D/(12-R)$. Solving gives R = 4 mph.

Prob 12. If a and b are integers greater than 0, then the number of pairs (a,b) that satisfy the equation $12a + b = ab$ is

a) 2 b) 4 c) 6 d) 8 e) none of these

12. solution. First note that a = 1 cannot satisfy the equation. Also note consecutive integers have no common divisor since and divisor of both would be a divisor of their difference and their difference is 1. Solving the given equation gives $b = 12a/(a-1)$. Since $a-1$ and a have no common divisor then $a-1$ must divide 12. Hence a must be 2, 3, 4, 5, 7 or 13 and the corresponding b is found from the equation. A less formal solution would be to try values of a in the equation $b = 12a/(a-1)$ and note that for a > 13, b is not an integer.

Prob. 13. If we call an integer boring if all of its digits are the same, then how many integers greater than 1 and less than 10000 are both boring and prime?

a) 4 b) 5 c) 7 d) 6 e) none of these

13. solution. All single digit positive integers are boring and the single digit primes, (2,3,5,7) are both prime and boring. All boring two digit positive integers are (11,22,33,44,55,66,77,88,99) and of these, only 11 is prime. The boring three digit positive integers are (111,222,….,999) and none of these are prime, since 111 = (3)(17). Similarly, since 1111 = (11)(101) and thus there are no 4 digit positive boring integers that are prime. Hence the only boring positive prime integers between 1 and 10000 are (2,3,5,7,11).

Prob. 14. One car goes around a lap in 40 seconds and a second car goes around the lap in 45 seconds. How many seconds will it take the faster car to gain one lap?

a) 300 b) 320 c) 340 d) 360 e) none of these

14. solution. In t seconds, the faster car will go t/40 laps and the slower will go t/45 laps. Hence $t/40 - t/45 = 1$ and t = 360.

Prob 15. Memory jog for 'geometric progression', 2, 4, 8, 16 is an example. If the sides of a right triangle form a geometric progression and the short side has length 1, then the hypotenuse has length

a) $(1+2\sqrt{5})/2$ b) $\sqrt{5}-1$ c) $(1+\sqrt{5})/2$ d) $2\sqrt{5}-1$ e) none of these

15. solution. If k is the common ratio of the progression a, b, c then

$$a^2+b^2=c^2 \quad \Rightarrow \quad a^2+(ak)^2=(ak^2)^2 \quad \Rightarrow \quad k^4-k^2-1=0.$$

Letting $z=k^2$ gives $z^2-z-1=0$ or $z=c=(1\pm\sqrt{5})/2$.

Prob. 16. The three boxes shown in the figure contain three positive integers (zero is not a positive integer). The three integers are all different and their sum is 8. How many different ways can the boxes be filled? (order counts, thus $a=2,\ b=5,\ c=1$ is not the same as $a=1,\ b=2,\ c=5$).

a	*b*	*c*

a) 9 b) 11 c) 13 d) 15 e) none of these

16. solution. If order does not count, the only possible triples are {1,2,5}, {1,3,4}. Each of these can be ordered in 6 distinct ways and thus there are 12 distinct ways to fill the boxes.

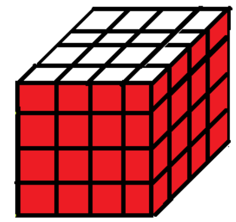
Prob. 17. The figure shows an addition problem with each letter representing a distinct digit. The leading digits are not zero. Find the digit represented by each letter. The sum of the digits F + R + S + X + T is

$$\begin{array}{r} \text{F O R T Y} \\ \text{T E N} \\ \underline{\text{T E N}} \\ \text{S I X T Y} \end{array}$$

a) 23 b) 24 c) 25 d) 26 e) none of th

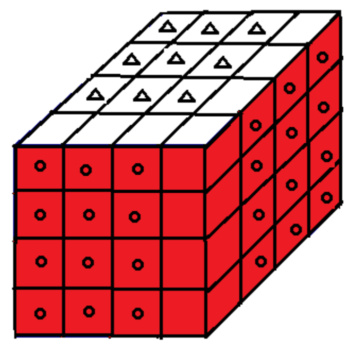
17. solution. N and E must be 0 or 5 and since there is no carry from the units digits addition then N = 0 and E = 5. O must be 8 or 9 since there must be a carry from the 1000s digit addition. If O = 8 then I = 0 but N is already 0, thus O = 9. Hence there is a carry of 2 from the 100s digit addition and I = 1. We now have [N,E,O,I] = [0,5,9,1]. Since S = F + 1 then [F,S] = [2,3], [3,4],[6,7] or [7,8]. As a result of the 10s and 100s digit carries, we have R + 2T +1 = 20 + X ≥ 22, since X cannot be 0 or 1. We seek triples [R,T,X] that satisfy the above conditions. If T < 6, then R $>$ 9 and this can't be. If T = 6 then [R,X] = [8,1] or [7,0] but these are both excluded

since 0 and 1 are already used . If T = 7 then R must be 6 or 8 and R = 6 is excluded, since that makes X = 1. There remains [8,7,3], [6,8,3] and [7,8,4] for possible values of [R,T,X]. If X = 3 then [F,S] must be [6,7] or [7,8] and thus we can exclude [8,7,3] and [6,8,3]. Thus [R,T,X] = [7,8,4] and this leaves [2,3] for [F,S] and Y must be the remaining digit 6. Hence F + R + S + X + T = 24.



Prob. 18. A cube has edges of length n, where n is an integer. The figure shows the cube with $n = 4$. Two faces with an edge in common are painted grey. The cube is then cut into n^3 smaller cubes with edge length 1. Let n_0, n_1 and n_2 be the number of cubes with exactly 0, 1 and 2 grey faces, respectively. If $n_0 / n_2 = 1600$ then n_1 is

a) 3280 b) 3360 c) 3240 d) 3320 e) none of these

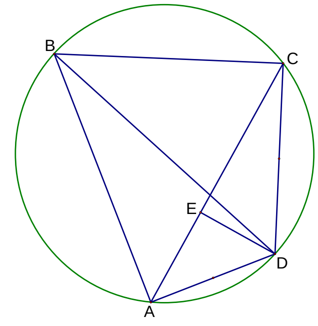


18. solution. The 24 small cubes with exactly one grey face are shown with the circles for the case $n = 4$. For $n = n$, there will be $n(n-1)$ on the front face of the large cube and the same on the right face, thus $n_1 = 2n(n-1)$. The 9 small cubes on the top slice, with no grey faces, are shown with triangles for the case $n = 4$. For $n = n$, the top slice will contain $(n-1)^2$ such cubes. Since there are n slices, then $n_0 = n(n-1)^2$. The cubes with exactly two grey faces will all be along the front right edge of the large cube and thus $n_2 = n$. Hence $n_0 / n_2 = (n-1)^2 = 1600$ and $n = 41$. Thus $n_1 = 2n(n-1) = 3280.$

19. I have twenty 3₵ stamps and twenty 5₵ stamps. Using one or more of these stamps, how many different amounts of postage can I make?

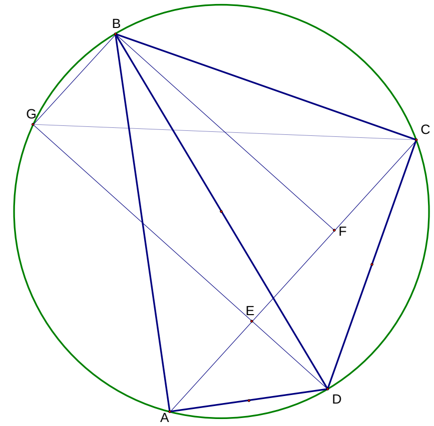
a) 148 b) 152 c) 163 d) 142 e) none of these

19. solution. If the postage amount is divided by 3, then the remainder will be 0, 1 or 2. Let A_0 be the set with remainder 0, A_1 be those with remainder 1 and A_2 be those with remainder 2. The set *A* = (3,6,…,60) is in A_0. The rest of A_0 is found by adding *k* of the 5₵ stamps to *A*, where *k* is a multiple of 3. If *k* = 18 we can make (93,96,…,150). The postage values (63,66,…,90) can be made using *k* = 12. Thus A_0 =(3,6,…,150) and 50 postal amounts can be made such that the remainder is 0 when divided by 3. The smallest amount in A_1 is 10 and similar calculations show that A_1 = (10,13,…160) giving 51 amounts. Finally A_2 = (5,8,…,155) also giving 51 amounts. Hence there are 152 different amounts of postage.



Prob. 20. Quadrilateral *ABCD* (shown, but not to scale) can be inscribed in a circle in such a way that *BD* is a diameter of the circle. Let *E* be the point on *AC* such that *AC* and *DE* are perpendicular. If $AE = 6$, $EC = 12$ and $DE = 5$, then the perpendicular distance from *B* to *AC* is

a) 13.8 b) 14.2 c) 14.4 d) 14.6 e) none of these



20. solution. Extend *DE* to its intersection *G* with the circle. Construct segment through *B* and perpendicular to *AC* at *F*. The quadrilateral *BGEF* is a rectangle since the angles at *F* and *E* are 90 degrees by construction. Also the angle at *G* is 90 degrees since *BD* is a diameter of the circle.

We now show that triangles *AED* and *GEC* are similar. The vertical angles at *E* are equal. The angle *EAD* and the angle *CGE* both subtend the arc *CD* on the circle and thus the two angles are equal. Hence triangles *AED* and *GEC* have equal corresponding angles and are similar. The ratios of corresponding sides are therefore equal and we have

$$\frac{ED}{EC} = \frac{AE}{GE} \quad \Rightarrow \quad \frac{5}{12} = \frac{6}{GE} \quad \Rightarrow \quad GE = BF = 14.4$$