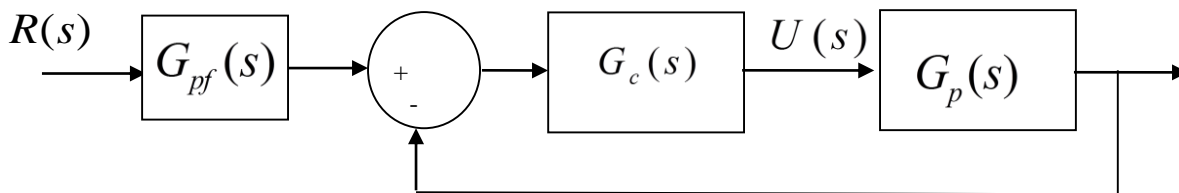


**ECE-320: Linear Control Systems**  
Homework 8

Due: Wednesday May 11 at the beginning of lab

1) One of the things that we need to worry about in practical systems is the limitation of the amplitude of the control signal, or the *control effort*. This is also a problem for most practical systems. In this problem we will do some simple analysis to better understand why Matlab's sisotool won't give us a good estimate of the control effort for some types of systems.

a) For the system below,



show that  $U(s)$  and  $R(s)$  are related by

$$U(s) = \frac{G_c(s)G_{pf}(s)}{1 + G_p(s)G_c(s)} R(s)$$

b) For many types of controllers, the maximum value of the control signal is just after the step is applied, at  $t = 0^+$ . Although most of the time we are concerned with steady state values and use the final value Theorem in the  $s$ -plane, in this case we want to use the initial value Theorem, which can be written as

$$\lim_{t \rightarrow 0^+} u(t) = \lim_{s \rightarrow \infty} sU(s)$$

If the system input is a step of amplitude  $A$ , show that

$$u(0^+) = \lim_{s \rightarrow \infty} \frac{AG_c(s)G_{pf}(s)}{1 + G_p(s)G_c(s)}$$

This result shows very clearly that the initial control signal is directly proportional to the amplitude of the input signal, which is pretty intuitive.

c) Now let's assume

$$G_c(s) = \frac{N_c(s)}{D_c(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)} \quad G_{pf}(s) = \frac{N_{pf}(s)}{D_{pf}(s)}$$

If we want to look at the initial value for a unit step, we need to look at

$$u(0^+) = \lim_{s \rightarrow \infty} \frac{sG_c(s)G_{pf}(s)}{1 + G_c(s)G_p(s)} \frac{1}{s} = \lim_{s \rightarrow \infty} \frac{G_c(s)G_{pf}(s)}{1 + G_c(s)G_p(s)}$$

Let's also then define

$$\tilde{U}(s) = \frac{G_c(s)G_{pf}(s)}{1 + G_c(s)G_p(s)}$$

so that

$$u(0^+) = \lim_{s \rightarrow \infty} \tilde{U}(s)$$

Show that

$$\tilde{U}(s) = \frac{N_{pf}(s)}{\left(\frac{D_c(s)}{N_c(s)}\right)D_{pf}(s) + \left(\frac{N_p(s)}{D_p(s)}\right)D_{pf}(s)}$$

and

$$\deg \tilde{U} = \deg N_{pf} - \max \left[ \deg D_c - \deg N_c + \deg D_{pf}, \deg N_p - \deg D_p + \deg D_{pf} \right]$$

where  $\deg Y$  is the degree of polynomial  $Y$ .

d) Since we are going to take the limit as  $s \rightarrow \infty$ , we need the degree of  $\tilde{U}(s)$  to be less than or equal to zero for a step input to have a finite  $u(0^+)$ . Why?

*For the remainder of this problem assume we have a 1 dof system such as an ideal second order system, so we have  $\deg N_p = 0$  and  $\deg D_p = 2$ .*

e) If the prefilter is a constant, show that in order to have a finite  $u(0^+)$  we must have

$$\deg D_c \geq \deg N_c$$

f) If the numerator of the prefilter is a constant, then in order to have a finite  $u(0^+)$  we must have

$$\deg D_c - \deg N_c + \deg D_{pf} \geq 0 \text{ or } -2 + \deg D_{pf} \geq 0$$

g) For P, I, D, PI, PD, PID, and lead controllers, determine if  $u(0^+)$  is finite if the prefilter is a constant.

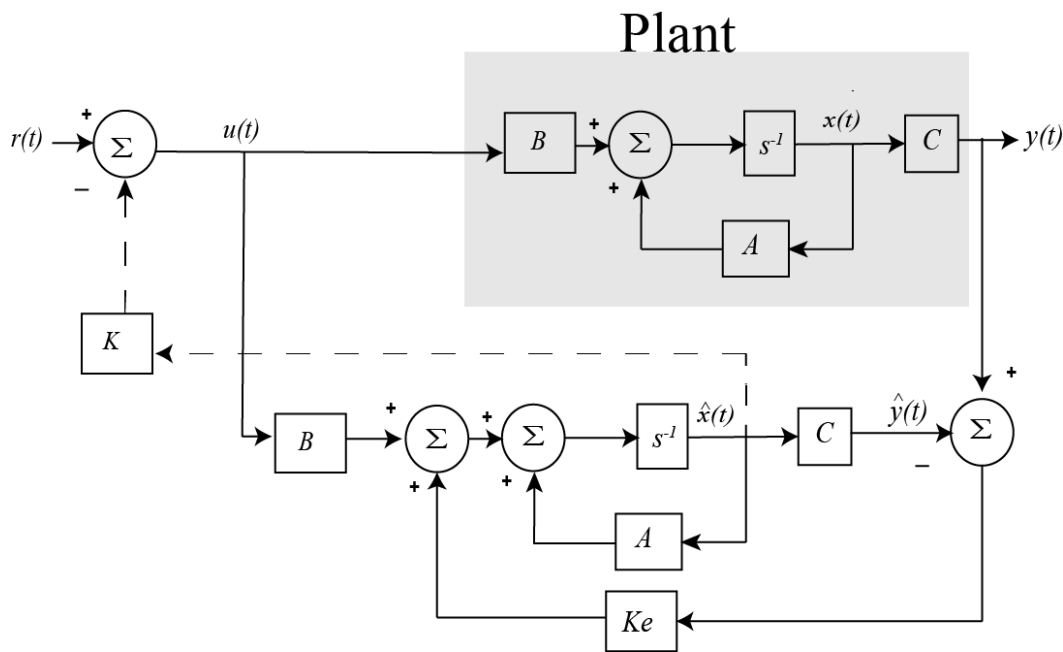
Note: Although it may appear that the control effort is sometimes infinite, in practice this is not true since our motor cannot produce an infinite signal. This large initial control signal is referred to as a *set-point kick*. There are different ways to implement a PID controller to avoid thi

For the following two problems we need the following information. If we have a state variable model with state feedback,  $\dot{x}(t) = Ax(t) + Bu(t)$   $u(t) = G_{pf}r(t) - Kx(t)$ , and we combine them, then we have  $\dot{x}(t) = [A - BK]x(t) + [BG_{pf}]r(t)$ . The poles of the system are determined by the eigenvalues of  $A - BK$ . The Matlab command **place** assumes we have a system in this form and we indicate the  $A$  and  $B$  matrices, indicate where we want the eigenvalues to be, and it returns  $K$ . So in the problems below, one of our goals is to put the system into this form.

Most of the answers to the following questions are very short.

2) In this problem we show how to use an observer to estimate the states of a feedback system so we do not have to measure them all. In this formulation, we only need a model of the plant, the input and the output (we need the input and the output for transfer function control too!)

Consider the following continuous-time state variable system with an observer:



For this system we have the plant representation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t)$$

and the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_e[y(t) - \hat{y}(t)] \quad \hat{y}(t) = C\hat{x}(t)$$

Note that the observer follows the same system dynamics as the plant (assuming we know the  $A$  and  $B$  matrices) plus a term that “correct” for a difference between the true output of the system  $y(t)$  and the estimated output from the observer  $\hat{y}(t)$  by an amount  $K_e$ . We use the estimated state  $\hat{x}(t)$  in the state variable feedback.

a) Assuming the output is a scalar and the state variable is an  $n \times 1$  vector, what is the size of  $K_e$ ?

b) Show that we can rewrite the observer equation as

$$\dot{\hat{x}}(t) = (A - K_e C)\hat{x}(t) + Bu(t) + K_e C x(t)$$

c) Show that if we subtract the equation in part (b) from the equation representing the plant we get

$$\dot{x}(t) - \dot{\hat{x}}(t) = [A - K_e C](x(t) - \hat{x}(t))$$

d) Now let's define the error vector as the difference between the true state and the estimated state,  $e(t) = x(t) - \hat{x}(t)$ , and with this definition show that the equation in part c is

$$\dot{e}(t) = [A - K_e C]e(t)$$

We want our estimate to equal the true state (eventually) so clearly we want  $e(t) \rightarrow 0$  as  $t \rightarrow \infty$  so we want to choose  $K_e$  so this happens.

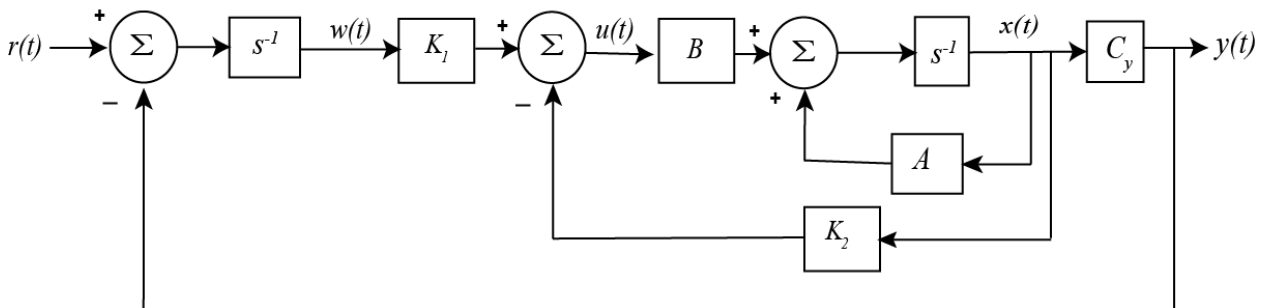
e) At this point we are basically done with everything we need to do except use the following little trick from linear algebra:  $(ABC)^T = C^T B^T A^T$ . With this trick show that we can write the above equations in the form

$$\dot{e}(t) = [\tilde{A} - \tilde{B}\tilde{K}] e(t)$$

and determine  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{K}$ .

**3)** In this problem we show how to make a state variable feedback system into a type one system.

Consider the following continuous-time state variable system:



For this system we have

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = C_y x(t)$$

$$u(t) = -K_2 x(t) + K_1 w(t)$$

$$\dot{w}(t) = r(t) - y(t)$$

- a) Show that we can represent the open loop system in the following form, and determine the  $\alpha_i$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} \alpha_5 \\ \alpha_6 \end{bmatrix} u(t) + \begin{bmatrix} \alpha_7 \\ \alpha_8 \end{bmatrix} r(t)$$

- b) Define

$$x_e(t) = x(t) - x(\infty)$$

$$w_e(t) = w(t) - w(\infty)$$

$$u_e(t) = u(t) - u(\infty)$$

and assume  $r(t) = r(\infty) = 1$ . Show that we can write this new system in the following form and determine the  $\beta_i$

$$\begin{bmatrix} \dot{x}_e(t) \\ \dot{w}_e(t) \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} x_e(t) \\ w_e(t) \end{bmatrix} + \begin{bmatrix} \beta_5 \\ \beta_6 \end{bmatrix} u_e(t)$$

- c) Show that we can write  $u_e(t) = -K_2 x_e(t) + K_1 w_e(t)$ . Don't just use the results from part a, you also need to use part b.

- d) Now define

$$e(t) = \begin{bmatrix} x_e(t) \\ w_e(t) \end{bmatrix}$$

Show that we can now write the equations

$$\dot{e}(t) = \hat{A}e(t) + \hat{B}u_e(t)$$

$$u_e(t) = -\hat{K}e(t)$$

What are the  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{K}$ ?

- e) Show that we can then write  $\dot{e}(t) = (\hat{A} - \hat{B}\hat{K})e(t)$  which is the standard form for pole placement of a state variable feedback system. Once we have found  $\hat{K}$  we can extract  $K_1$  and  $K_2$