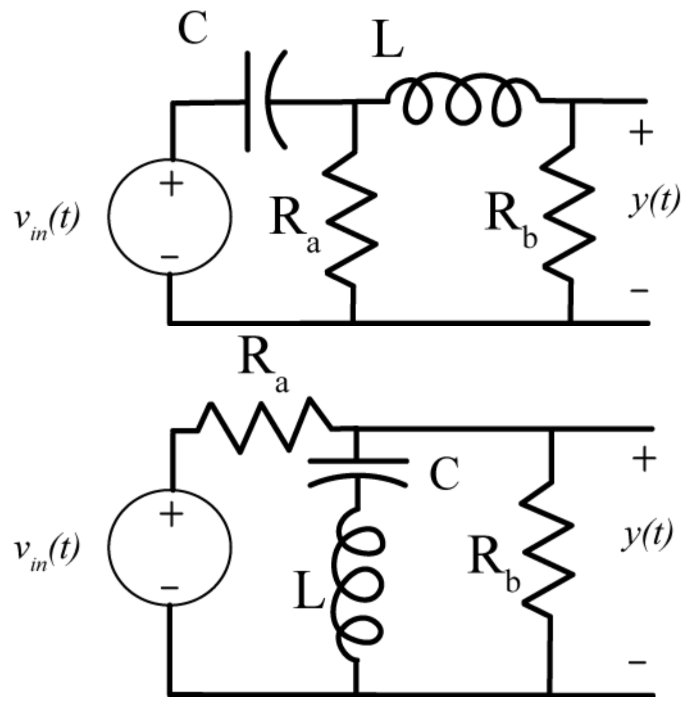
ECE-320: Linear Control Systems
Homework 7

Due: Monday May 2 at the beginning of class
Exam 2, Friday May 6

1) For the following two circuits,

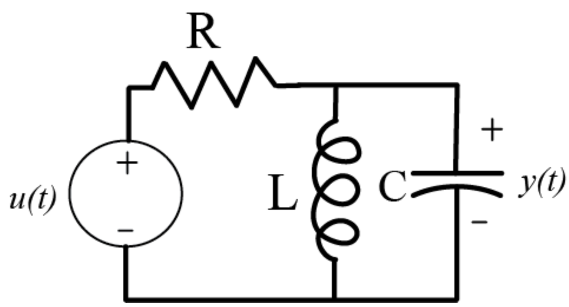


show that the state variable descriptions are given by

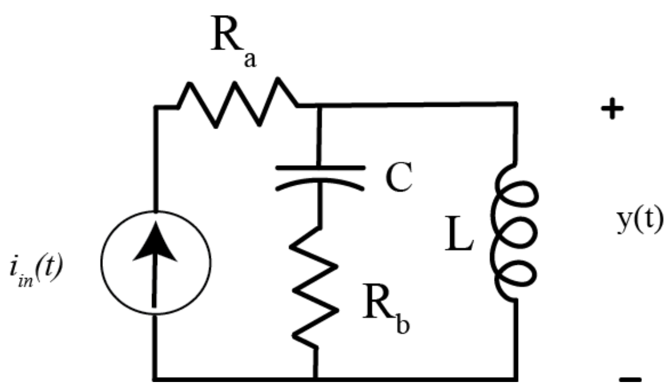
$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_b}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_a C} \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_a C} \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} R_B & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + [0] v_{in}(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a R_b}{L(R_a+R_b)} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{L(R_a+R_b)} \\ 0 \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} -\frac{R_a R_b}{R_a+R_b} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{R_a+R_b} \end{bmatrix} v_{in}(t)$$

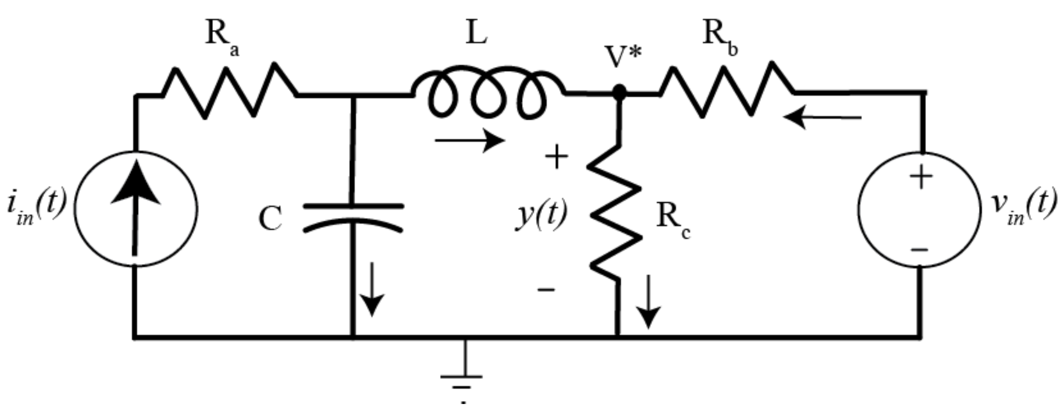
2) For the following circuit, the state variables are the current through the inductor and the voltage across the capacitor. Determine a state variable model for this system. Specifically, you need to identify the A, B, C, and D matrices/vectors/scalars.



3) For the following circuit, the state variables are the current through the inductor and the voltage across the capacitor. Determine a state variable model for this system. Specifically, you need to identify the A, B, C, and D matrices/vectors/scalars.



4) For the following circuit, the state variables are the current through the inductor and the voltage across the capacitor. Determine a state variable model for this system. Specifically, you need to identify the A, B, C, and D matrices/vectors/scalars. For my sanity, and to get full credit, you should assume the currents flow in the direction shown by the arrows. Also, you may find it easier to use the node voltage v* and then figure out what it is in terms of state variable. Also note that there are two inputs in this system.



5) For the plant $G_p(s) = \dfrac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$

a) If the plant input is $u(t)$ and the output is $x(t)$, show that we can represent this system with the differential equation

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = K\omega_n^2 u(t)$$

b) Assuming we use states $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$, and the output is $x(t)$, show that we can write the state variable description of the system as

$$\frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t) \quad y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form $u(t) = G_{pf}r(t) - kq(t)$, where $r(t)$ is the new input to the system, G_{pf} is a prefilter (for controlling the steady state error), and k is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$
$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$
$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1}BG_{pf} + DG_{pf}$$

and if D is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1}BG_{pf}$$

e) Assume $r(t) = u(t)$ and $D = 0$. Show that, in order for $\lim_{t\to\infty} y(t) = 1$, we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1}B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix P *is given as*

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of P *is given by* $ad-bc$*. This determinant will also give us the characteristic polynomial of the system.*

6) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if k_1 and k_2 exist ($k = [k_1 \quad k_2]$) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like $s^2 + a_1 s + a_0$ for any a_1 and any a_0. If this is true, the system is said to be ***controllable***.

a) Show that for

$$\dot{q} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$

b) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{sG_{pf}}{s^2 + (k_2 - 1)s + k_1}$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{G_{pf}}{s^2 + (k_2 - 1)s + (k_1 - 1)}$