ECE-320: Linear Control Systems
Homework 9

Due: Tuesday November 1 at 2 PM

1) Consider the plant

$$G_p(s) = \frac{\alpha_0}{s+\alpha_1} = \frac{3}{s+0.5}$$

where 3 is the nominal value of α_0 and 0.5 is the nominal value of α_1. In this problem we will investigate the sensitivity of closed loop systems with various types of controllers to these two parameters. We will assume we want the settling time of our system to be 0.5 seconds and the position error for a unit step input to be less than 0.1.

a) (*ITAE Model Matching*) Since this is a first order system, we will use the first order ITAE model,

$$G_o(s) = \frac{\omega_o}{s+\omega_o}$$

i) For what value of ω_o will we meet the settling time requirements and the position error requirements?

ii) Determine the corresponding controller $G_c(s)$.

iii) Show that the closed loop transfer function (using the parameterized form of $G_p(s)$ and the controller from part ii) is

$$G_o(s) = \frac{\frac{8}{3}\alpha_0(s+0.5)}{s(s+\alpha_1)+\frac{8}{3}\alpha_0(s+0.5)}$$

iv) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s}{s+8}$$

v) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+8.5s+4}$$

b) (*Quadratic Optimal*) For $q = 7.083$, show that the quadratic optimal closed loop transfer function is approximately

$$G_o(s) = \frac{8}{s+8}$$

Hence the results of both of our model matching methods will be very nearly the same.

c) (*Proportional Control*) Consider a proportional controller, with $k_p = 2.5$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{2.5\alpha_0}{s+\alpha_1+2.5\alpha_0}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s+0.5}{s+8}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5}{s+8}$$

d) (*Proportional+Integral Control*) Consider a PI controller with $k_p = 4$ and $k_i = 40$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{4\alpha_0(s+10)}{s(s+\alpha_1)+4\alpha_0(s+10)}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s(s+0.5)}{s^2+12.5s+120}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+12.5s+120}$$

e) (*Diophantine Equations*) Consider a controller designed using the Diophantine equations. Assume we want a type 1 system and closed loop poles both at -8.

i) Show that the controller is given by

$$G_c(s) = \frac{21.333 + 5.1667s}{s}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s(s+0.5)}{(s+8)^2}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{(s+8)^2}$$

f) Using Matlab, simulate the unit step response of each type of controller (except the quadratic optimal). Plot all responses on one graph. Use different line types and a legend. Turn in your plot and code.

g) Using Matlab and subplot, plot the sensitivity to α_0 for each type of controller (except the quadratic optimal) on ***one graph*** at the top of the page, and the sensitivity to α_1 on one graph on the bottom of the page. Be sure to use different line types and a legend. Turn in your plot and code. Only plot up to about 8 Hz (50 rad/sec) using a semilog scale with the sensitivity in dB (see below). ***Do not*** *make separate graphs for each system!*

In particular, these results should show you that the model matching methods, which essentially try and cancel the plant, are generally more sensitive to getting the plant parameters correct that either the PI or the Diophantine methods for low frequencies. However, for higher frequencies the methods are all about the same. The P controller, the simplest of the four, is always the worst.

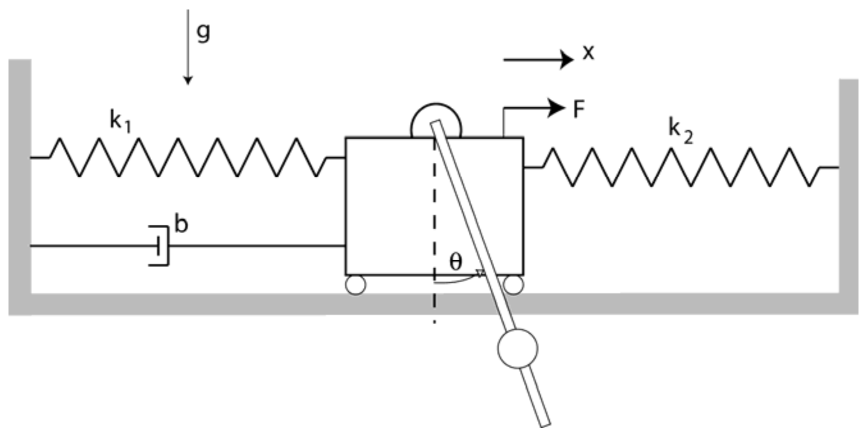
Hint: If $T(s) = \frac{2s}{s^2 + 2s + 10}$, *plot the magnitude of the frequency response using:*

```
T = tf([2 0],[1 2 10]);
w = logspace(-1,1.7,1000);
[M,P]= bode(T,w);
Mdb = 20*log10(M(:));
semilogx(w,Mdb); grid;
xlabel('Frequency (rad/sec)');
ylabel('dB');
```

Preparation for Lab 9 (No Maple)

2) *In this derivation we will make a state variable model for a regular pendulum (a pendulum hanging down) attached to the first cart. It will be easier to measure the parameters for a regular pendulum since it is a stable system. In the lab we will initially try and control the regular pendulum. Once this is working, we will try to control an inverted pendulum. To go from the model of a regular pendulum to the model of an inverted pendulum we use the substitution* $l \rightarrow -l$.

Consider the configuration for the regular pendulum shown below:



The equations of motion for the regular pendulum can be written as

$$\begin{aligned} (J+ml^2)\ddot{\theta} + ml\cos(\theta)\ddot{x} + mgl\sin(\theta) &= 0 \\ (M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) + c\dot{x} + kx &= F \end{aligned}$$

The mass of the cart is M, the mass at the center of mass of the pendulum is m, the moment of inertial of the pendulum about its center of mass is J, L is the length of the pendulum, and l is the distance from the pivot to the center of mass of the pendulum. The angle θ is measured counterclockwise from straight up, x is the displacement of the first cart (positive to the right), and g is the gravitational constant.

a) Using a small angle/small velocity assumption, show that we can approximate the above equations of motion as

$$\begin{aligned} (J+ml^2)\ddot{\theta} + ml\ddot{x} + mgl\theta &\approx 0 \\ (M+m)\ddot{x} + ml\ddot{\theta} + c\dot{x} + kx &\approx F \end{aligned}$$

b) We can rewrite the first equation above as

$$\frac{1}{\omega_\theta^2}\ddot{\theta} + \frac{1}{g}\ddot{x} + \theta = 0$$

What is ω_θ^2?

c) If we assume the cart is fixed, then $\ddot{x}=0$ and we have

$$\ddot{\theta}+\omega_\theta^2\theta=0$$

This is the equation for a simple pendulum. If the pendulum is deflected a small angle and released, it will oscillate with frequency ω_θ. If we measure the period of the oscillations T_θ how do we find ω_θ?

d) We can rewrite the second equation from (a) as

$$\frac{1}{\omega_1^2}\ddot{x}+\frac{2\zeta_1}{\omega_1}\dot{x}+x+K_1\ddot{\theta}=K_2F$$

Find expressions for ω_1, ζ_1, K_1, and K_2 in terms of m, M, k, c, and l. If we assume there is no input ($F=0$) and the pendulum does not move very much ($\ddot{\theta}\approx 0$) then we can use the log-decrement method to get initial estimates of ω_1 and ζ_1.

e) Assuming we apply a step input of amplitude A to the cart, show that in steady state we get

$$K_2=\frac{x_{ss}}{A}$$

f) Show that

$$\frac{\Theta(s)}{X(s)}=-\frac{\omega_\theta^2}{g}\left(\frac{s^2}{s^2+\omega_\theta^2}\right)$$

We need to measure the gravitational constant in cm, since all other distances are measured in cm.

g) Show that

$$\frac{X(s)}{F(s)}=\frac{\omega_1^2K_2(s^2+\omega_\theta^2)}{\left(1-K_1\omega_1^2\frac{\omega_\theta^2}{g}\right)s^4+\left(2\zeta_1\omega_1\right)s^3+(\omega_1^2+\omega_\theta^2)s^2+(2\zeta_1\omega_1\omega_\theta^2)s+\omega_1^2\omega_\theta^2}$$

We can use this expression to determine K_1 and get better estimates of ω_1 and ζ_1.

h) We can rewrite our linearized dynamical equations as

$$\begin{aligned}\ddot{\theta} &\approx -\frac{1}{g}\omega_\theta^2\ddot{x} - \omega_\theta^2\theta \\ \ddot{x} &\approx -2\zeta_1\omega_1\dot{x} - \omega_1^2 x - K_1\omega_1^2\ddot{\theta} + K_2\omega_1^2 F\end{aligned}$$

By substituting the second equation into the first equation, show that we get

$$\ddot{\theta} \approx \frac{1}{\Delta}\left(\frac{\omega_1^2\omega_\theta^2}{g}\right)x + \frac{1}{\Delta}\left(\frac{2\zeta_1\omega_1\omega_\theta^2}{g}\right)\dot{x} + \frac{1}{\Delta}\left(-\omega_\theta^2\right)\theta + \frac{1}{\Delta}\left(-\frac{\omega_1^2\omega_\theta^2 K_2}{g}\right)F$$

where

$$\Delta = 1 - \left(\frac{K_1\omega_1^2\omega_\theta^2}{g}\right)$$

and substituting the first equation into the second equation we get

$$\ddot{x} \approx \frac{1}{\Delta}\left(-\omega_1^2\right)x + \frac{1}{\Delta}\left(-2\zeta_1\omega_1\right)\dot{x} + \frac{1}{\Delta}\left(K_1\omega_1^2\omega_\theta^2\right)\theta + \frac{1}{\Delta}\left(K_2\omega_1^2\right)F$$

i) Defining $q_1 = x$, $q_2 = \dot{x}$, $q_3 = \theta$, and $q_4 = \dot{\theta}$, show that we get the following state equations

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -\left(\frac{\omega_1^2}{\Delta}\right) & -\left(\frac{2\zeta_1\omega_1}{\Delta}\right) & \left(\frac{K_1\omega_1^2\omega_\theta^2}{\Delta}\right) & 0\\ 0 & 0 & 0 & 1\\ \left(\frac{\omega_1^2\omega_\theta^2}{g\Delta}\right) & \left(\frac{2\zeta_1\omega_1\omega_\theta^2}{g\Delta}\right) & -\left(\frac{\omega_\theta^2}{\Delta}\right) & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + \begin{bmatrix}0\\ \left(\frac{K_2\omega_1^2}{\Delta}\right)\\ 0\\ -\left(\frac{\omega_1^2\omega_\theta^2 K_2}{g\Delta}\right)\end{bmatrix}F$$

j) If we now want to model the inverted pendulum ($l \to -l$ to go back to the inverted pendulum*)*, which terms change in the matrices above?

k) When we try and fit the frequency response data we see in the lab we will often get an unusual response. To understand this response we will analytically try and show what is happening. If you've not screwed up, you should have obtained values of

$$\begin{aligned}K_1 &= \frac{ml}{k} \\ \omega_1^2 &= \frac{k}{M+m} \\ \omega_\theta^2 &= \frac{mgl}{J+ml^2}\end{aligned}$$

If we assume that the mass of the cart and pendulum attachment is much larger than the mass at the center of mass of the pendulum, then we have M is much larger than m. Secondly, J is the moment of inertia about the center of mass of the pendulum, m is the mass at the center of mass of the pendulum, and l is the distance from the pivot to the center of mass of the pendulum. For our systems, the pendulum bars have negligible mass and all of the mass is essentially concentrated at the center of mass. Hence ml^2 is much larger than J. Using these two assumptions, show that $\Delta \approx 1$.

l) Assuming $\Delta \approx 1$, show that

$$\frac{X(s)}{F(s)} \approx \frac{\omega_1^2 K_2}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2}$$

That is, there is a pole/zero cancellation! This is the effect you will often see in lab.

3) Modify ***Basic_2dof_State_Variable_Model.mdl,*** *and* ***Basic_2dof_State_Variable_Model_Driver.m*** to work with the ***regular*** pendulum model. (These files and the state model for the regular pendulum are available one the course website.) Specifically, you need to

- Modify the code to access the correct states. Specifically, set
  get_desired_states = [1 0 0 0 0 0 0 0;
  0 1 0 0 0 0 0 0;
  0 0 0 0 0 0 1 0;
  0 0 0 0 0 0 0 1];
- Rename the output states to be m_x1, m_x1_dot, m_theta, m_theta_dot
- Plot the position of the cart, the velocity of the cart, the position of the pendulum, and the velocity of the pendulum, the system output (probably the position of the pendulum), and the control effort. All plots should be neatly organized on one page.
- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- The output of the system is the position of the pendulum, so modify the C vector accordingly.
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.
- Utilize the linear quadratic regulator or pole placement method to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight down and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4 and the system should come to steady state in less than 1.0 seconds.

You will need to turn in you plot, your Simulink code, and your Matlab code.

4) Utilize the results of problem 3 to model the ***inverted*** pendulum. The only thing you should need to change is the state model. (The state model is available one the course website.) Specifically, you need to

- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.

- Utilize the linear quadratic regulator or pole placement method to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight up and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4. Limiting the cart motion is usually the most difficult part. Often your controller for the regular pendulum will work for this part too, but not always.

You will need to turn in your plot for this part.