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ECE-205

Exam 1

Winter 2009

Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1 ________

Problem 2 ________

Problem 3 ________

Problem 4 ________

Problem 5 ________

Total ____________

Results	90-100	7
	80-89	1
	70-79	2
	60-69	0
	<60	4

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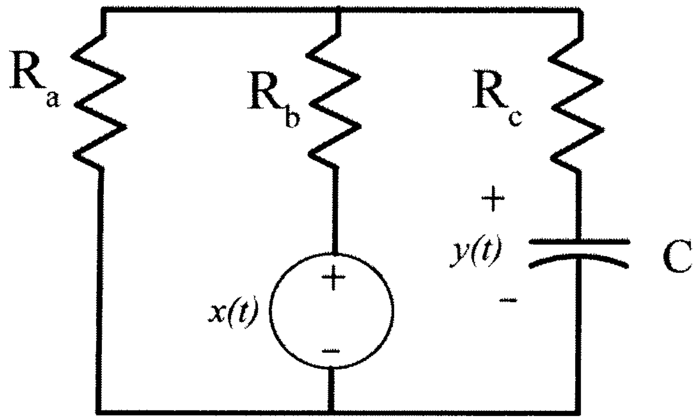
1) (15 points) Derive the governing differential equation for the following first order circuit. You can use any method you want (except for copying). You do not need to put your answer in a standard form.



From Capacitor $\boxed{R_{th} = \underbrace{R_c + R_a \| R_b}_{\text{good enough}}} = R_c + \frac{R_a R_b}{R_a + R_b} = \frac{R_c R_a + R_c R_b + R_a R_b}{R_a + R_b}$

$\boxed{\tau = R_{th} C}$ $\quad y(\infty) = \frac{R_a}{R_a + R_b} x(\infty)$ $\quad \boxed{K = \frac{R_a}{R_a + R_b}}$

$$\boxed{\tau \dot{y} + y = Kx}$$

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2) (30 points) Assume we have a first order system with the governing differential equation

$$2\dot{y}(t)+4y(t)=8x(t)$$

The system is initially at rest, so $y(0)=0$. The input to this system is

$$x(t)=\begin{cases}0 & t\le 0\\ 3 & 0<t\le 1\\ -2 & 1<t\le 1.5\\ 1 & 1.5<t\end{cases}$$

Determine the output of the system in each of the above time intervals. *Simplify your final answer as much as possible and box it.*

$\frac{1}{2}\dot{y}+y=2x \qquad \tau=\frac{1}{2} \quad K=2 \qquad y(t)=[y(0)-y(\infty)]e^{-t/\tau}+y(\infty)$

① $t\le 0 \quad y(t)=0$

① $0\le t\le 1 \quad y(0)=0 \quad y(\infty)=KA=(2)(3)=6$

$y(t)=[0-6]e^{-2t}+6=6(1-e^{-2t})$

② $1\le t\le 1.5 \quad "y(0)"=y(1)=6(1-e^{-2})=5.188 \quad "y(\infty)"=KA=2(-2)=-4$

$y(t)=[5.188-(-4)]e^{-2(t-1)}-4=9.188e^{-2(t-1)}-4$

③ $t\ge 1.5 \quad "y(0)"=y(1.5)=9.188e^{-1}-4=-0.620 \quad "y(\infty)"=KA=(2)(1)=2$

$y(t)=[-0.620-2]e^{-2(t-1.5)}+2=-2.620e^{-2(t-1.5)}+2$

$$\boxed{y(t)=\begin{cases}0 & t\le 0\\ 6(1-e^{-2t}) & 0\le t\le 1\\ 9.188e^{-2(t-1)}-4 & 1\le t\le 1.5\\ -2.620e^{-2(t-1.5)}+2 & t\ge 1.5\end{cases}}$$

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3) (15 points) Assume we have a first order system with the governing differential equation

$$2\dot{y}(t) + 3y(t) = e^{-t}x(t-1)$$

The initial time is $t_0 = 1$ and initial value $y(1) = 2$. Use the method of integrating factors to determine the output $y(t)$ as a function of the (unknown) input $x(t)$. *Simplify your answer as much as possible and box it.*

$$\dot{y}(t) + \frac{3}{2}y(t) = \frac{1}{2}e^{-t}x(t-1)$$

$$\frac{d}{dt}\left[y(t)e^{\frac{3}{2}t}\right] = \frac{1}{2}e^{\frac{3}{2}t}e^{-t}x(t-1) = \frac{1}{2}e^{\frac{t}{2}}x(t-1)$$

$$\int_{t_0}^{t}\frac{d}{d\lambda}\left[y(\lambda)e^{\frac{3}{2}\lambda}\right]d\lambda = y(t)e^{\frac{3}{2}t} - y(t_0)e^{\frac{3}{2}t_0} = \int_{t_0}^{t}\frac{1}{2}e^{\frac{\lambda}{2}}x(\lambda-1)d\lambda$$

$$y(t) = y(t_0)e^{-\frac{3}{2}(t-t_0)} + e^{-\frac{3}{2}t}\frac{1}{2}\int_{t_0}^{t}e^{\frac{\lambda}{2}}x(\lambda-1)d\lambda$$

$$\boxed{y(t) = 2e^{-\frac{3}{2}(t-1)} + \frac{1}{2}e^{-\frac{3}{2}t}\int_{1}^{t}e^{\frac{\lambda}{2}}x(\lambda-1)d\lambda}$$

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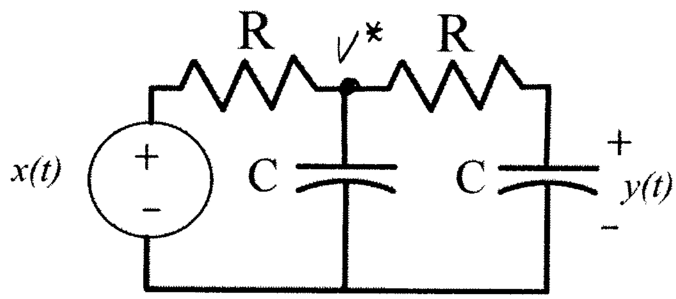
4) (20 points) For the second order circuit below, derive the governing second order differential equation for the output $y(t)$ and input $x(t)$. You do not need to put it into a standard form.



$$\frac{x - V^*}{R} = C\dot{V}^* + C\dot{y} \qquad V^* - RC\dot{y} = y$$

$$V^* = RC\dot{y} + y$$

$$x - V^* = RC\dot{V}^* + RC\dot{y}$$

$$x = V^* + RC\dot{V}^* + RC\dot{y}$$

$$= (RC\dot{y} + y) + RC(RC\ddot{y} + \dot{y}) + RC\dot{y}$$

$$= \boxed{(RC)^2\ddot{y} + 3RC\dot{y} + y = x}$$

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5) (20 points) The form of the under damped ($0 < \zeta < 1$) solution to the second order differential equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \mathrm{K}\ \omega_n^2 x(t)$$

for a step input $x(t) = Au(t)$ is

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where c and ϕ are constants to be determined and the damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

a) Using the initial condition $\dot{y}(0) = 0$ show that $\tan(\phi) = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

$$\dot{y}(t) = c(-\zeta\omega_n)e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) + ce^{-\zeta\omega_n t}(\omega_d)\cos(\omega_d t+\phi)$$

$$\dot{y}(0) = -\zeta\omega_n\sin(\phi) + \omega_d\cos(\phi) = 0$$

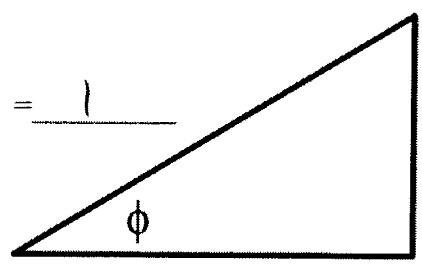
$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta}$$

$$\boxed{\tan(\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}}$$

b) We can express the relationship in part a using the following triangle. Fill in the blanks and then use this triangle determine an expression for $\sin(\phi)$.

$r^2 = x^2 + y^2$
$= \zeta^2 + 1 - \zeta^2 = 1$

r = 1



y = $\sqrt{1-\zeta^2}$

x = ζ

$$\boxed{\sin(\phi) = \sqrt{1-\zeta^2}}$$

c) Use your answer to part b, and the initial condition $y(0) = 0$ to determine the remaining unknown constant, and write out the complete solution for $y(t)$.

$$y(0) = 0 = KA + c\sin(\phi) = KA + c\sqrt{1-\zeta^2} \qquad c = \frac{-KA}{\sqrt{1-\zeta^2}}$$

$$\boxed{y(t) = KA\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right) \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$