Session #

Parametric Time Domain System Identification of a Mass-Spring-Damper System

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Abstract

One of the key objectives of any undergraduate system dynamics curriculum is to foster in the student an understanding of the limitations of linear, lumped parameter models. That is, the student must come face to face with the fact that models do not perform exactly like the physical system they are created to emulate. This is best done in the laboratory with a physical system that has small non-linearities which prevent the student from obtaining an exact match between model and experiment. This work describes an experiment designed for the sophomore system dynamics course offered at the Rose-Hulman Institute of Technology. This lab uses a commercially available hardware system and a digital computer. By a clever combination of various response data, and using known differences between effective masses, the effective inertia of motor, pinion, rack and cart are estimated without requiring disassembly of the system. Typical results are shown.

Introduction

The mechanical engineering and electrical engineering faculty at Rose-Hulman (RHIT) are currently upgrading the system dynamics and controls laboratory. One of the primary courses this lab services is Analysis and Design of Engineering Systems (ES 205) which is a sophomore level system dynamics course taught to all mechanical, electrical, and biomedical engineering majors. ES 205 focuses on lumped parameter modeling of mechanical, electrical, fluid, and thermal systems.

The hardware plant used in this lab is the Educational Control Products (ECP) Rectilinear Control System[1], shown in Figure 1. This is a translational mass-spring-damper system driven by a DC electric motor that provides up to three degrees of freedom of motion. System stiffnesses may be changed to the user's liking. A variable air damper may be connected to any of the masses.The plants also provide for varying the system mass by adding or removing 500g masses. Thus the *differences* in mass between possible configurations are well known.

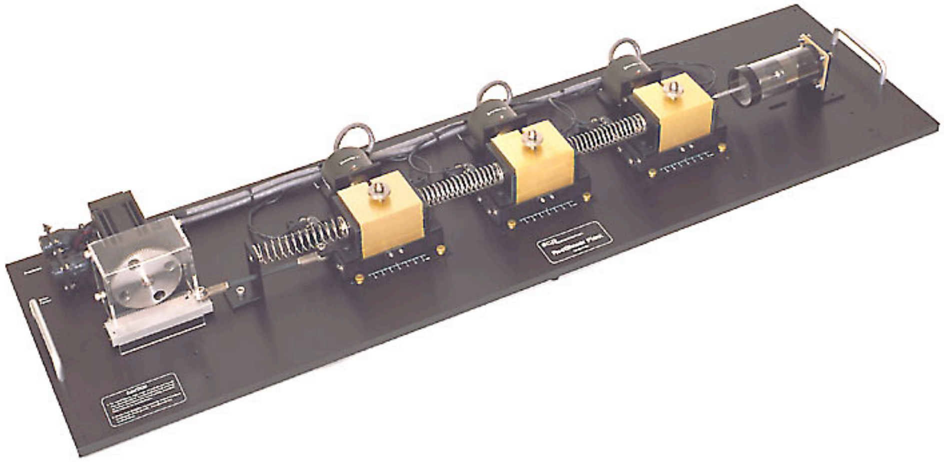

Fig. 1. The ECP Rectilinear Control System

Section 2 describes the problem and outlines the theory underlying parametric system identification, and the data reduction methods needed for this lab. Section 3 outlines the lesson objectives in detail. Section 4 discusses the data collection procedure. Section 5 presents typical results.

Problem Statement and Theory

One of the primary misconceptions that students fall into is that lumped parameter models can exactly predict the behavior of real systems. For example, Figure 2 shows a lumped parameter schematic of a single mass system.

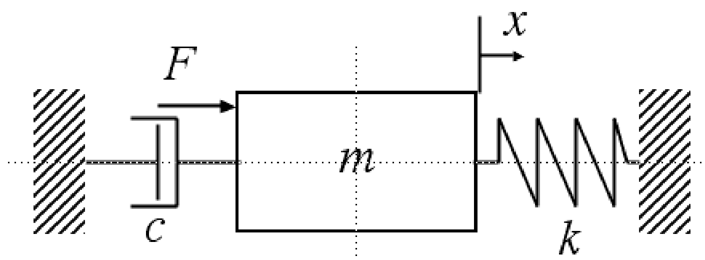

Fig. 2. Schematic of Lumped Parameter Model

The spring and damper are assumed to contribute no mass to the system. An electric motor is assumed to provide a force directly to the mass. A parametric differential equation model matching this schematic is given in Equations 1 and 2.

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}Kf(t) \tag{1}$$

$$\frac{m}{k}\ddot{x} + \frac{c}{k}\dot{x} + x = Kf(t). \tag{2}$$

The model and experimental step responses are compared in Figure 3. Although the match is fairly close, it will never be exact due to small non-linearities in the system.

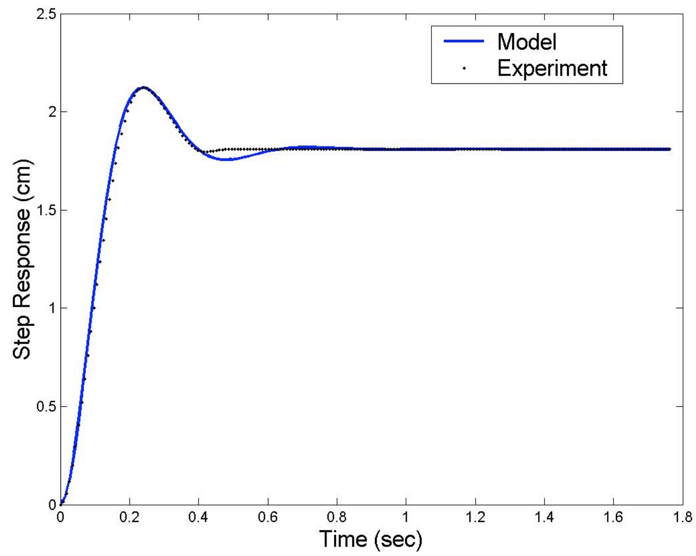

Fig. 3. Comparison of Expermental and Parametric Model Step Response *m* = 0.9538kg, *k* = 216.2 N/m, *c* = 14.01 N-s/m

The parametric models in Eqns 1 and 2 can be directly compared to their non parametric counterparts

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = K\omega_n^2 f(t) \tag{3}$$

and

$$\frac{\ddot{x}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{x} + x = Kf(t)\,. \tag{4}$$

Matching zeroth order coefficients between Eq. 1 and Eq. 3 or second order coefficients between Eq. 2 and Eq. 4, we obtain the familiar formula for system natural frequency.

$$\omega_n^2 = \frac{k}{m} \tag{5}$$

Combining this with the definition of damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, we arrive at the following equation relating damping ratio, stiffness, mass, and damped natural frequency.

$$\omega_{di} = \sqrt{\frac{k}{m_i}}\sqrt{1-\zeta_i^2}$$

The subscript *i* indicates that we will use several values of mass, but a constant stiffness.
These eqns can be manipulated to the following form which is linear in the unknowns (*k* and m_i)

$$m_i \omega_{di}^2 - k\left(1 - \zeta_i^2\right) = 0 \tag{6}$$

The values of damped natural frequency (ω_{di}) are obtained by measuring the response peak time and applying the relationship

$$t_p = \frac{\pi}{\omega_d}. \tag{7}$$

The values of damping ratio are found by measuring the peak response (y_{peak}) and steady state response (y_{ss}) and computing the decimal form of percent overshoot (M_p).

$$M_p = \frac{y_{peak} - y_{ss}}{y_{ss}}$$

Percent overshoot is related to damping ratio by the formula

$$M_p = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right).$$

Which can be inverted, yielding

$$\zeta = \sqrt{\frac{\left(\ln\left(M_p\right)\right)^2}{\left(\ln\left(M_p\right)\right)^2 + \pi^2}}. \tag{8}$$

From their coursework in ES 205, students are familiar with the typical second order response characteristics such as settling time, percent overshoot, and peak time, and how these characteristics relate to non-parametric model properties, namely natural frequency and damping ratio. Homework exercises include determining peak time and percent overshoot from a typical experimental step response such as that shown in Figure 3, and determining a second order non-parametric model from these response characteristics. However, without knowing at least one of the parameters *m*, *c* or *k* of Eqs. 1 and 2 from physical measurements, it is impossible to uniquely determine the three system physical parameters from the two properties of natural frequency and damping ratio. This experiment is designed to take the student deeper by combining information from several step responses.

For the step response of cases four through seven shown in Table 1 below, the student is required to determine the peak time and percent overshoot, then apply Eqs. 7 and 8 to find the corresponding damped natural frequency and damping ratio. These values of damped natural frequency and damping ratio can then be used in Eq. 6 to form four independent equations in the five unknowns m_i, i=1,2,3,4, and k. Three additional eqns are formed from the known difference between masses

$$
\begin{aligned}
m_{i+1} - m_i &= 0.5, \quad i = 1,3 \\
m_{i+1} - m_i &= 1.0, \quad i = 2
\end{aligned}
\tag{9}
$$

The seven equations described above form the overdetermined system

$$
\begin{bmatrix}
\omega_{d1}^2 & 0 & \cdots & \left(\zeta_1^2 - 1\right) \\
-1 & 1 & \cdots & 0 \\
0 & \omega_{d2}^2 & \cdots & \left(\zeta_2^2 - 1\right) \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ k \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0.5 \\ 0 \\ \vdots \end{bmatrix}.
\tag{10}
$$

Solving the system in Eq. 10, results in estimates of the four masses and common stiffness used for the four step responses.

The damping constant c can then be found by matching coefficients between the expected system model and standard forms of the model. That is, matching coefficients between the parametric model of Eq. 1 and the form based on second-order response characterstics in Eq. 3 results in the following formula for the damping constant c.

$$c = 2\zeta m \frac{\omega_d}{\sqrt{1 - \zeta^2}}$$

Matching coefficients between the parametric form of Eq. 2 and the Bode form of the second order ODE given in Eq. 4 results in the formula

$$c = \frac{2\zeta k \sqrt{1 - \zeta^2}}{\omega_d}$$

Lesson Objectives

In this lab, we seek to identify the parameters of a one degree of freedom mass-spring-damper system. After completing this experiment, the student should be able to identify the effective mass, stiffness, and damping of a system from measured response data. Although we wish to ignore motor dynamics, because of the direct connection from motor, to pinion, rack, and subsequently, mass, we must account for the inertia of these components in our model. Also, the damper contributes mass which must be lumped in as well. The effective mass includes the inertial effects of all system elements including, for example, the mass of the damping and spring elements which are typically neglected in ideal textbook problems.

Data Collection

The student is required to take step response data for the system varying the stiffness, mass and damping as shown in Table 1. Cases four through seven hold the stiffness and damping constant, and require incremental mass changes for the solution of Eqn 10 above. The other cases are intended to illustrate the direct effects of independently varying damping and stiffness.

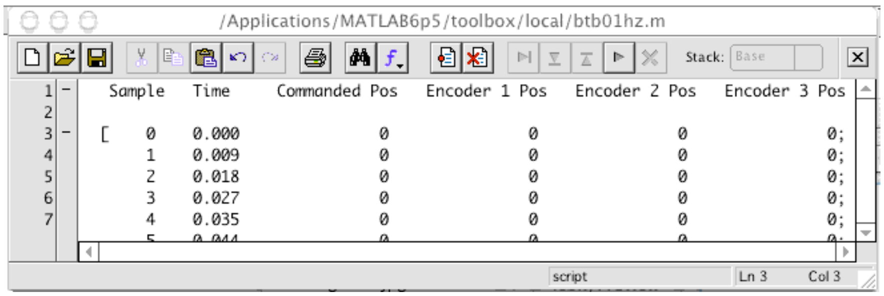

Fig. 4. ECP Data Export Format

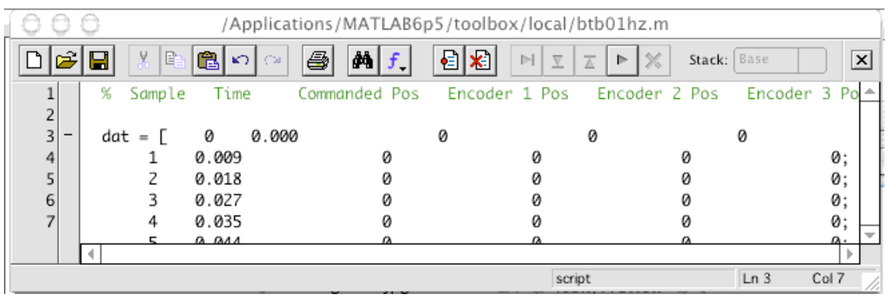

Fig. 5. Data File After Editing

The ECP software exports the data in space delimited columns which are readily imported into Matlab. Figures 4 and 5 show respectively the typical data file format and how it can be easily edited to be an executable Matlab script which loads the entire data set.

Table 1. Test Matrix

Case	Added Mass	Damper	Spring
1	1000g,	1/2 turn,	Medium
2	1000g,	4 full turns	Medium
3	1000g	no damper	Medium
4	0g (Cart Empty)	1/2 turn	Medium
5	500g	1/2 turn	Medium
6	1500g	1/2 turn	Medium
7	2000g	1/2 turn	Medium
8	1000g	1/2 turn	Stiff
9	1000g	1/2 turn	Light

The system is equipped with a removable and adjustable air damper. Case 1 is considered the benchmark against which all other cases are compared. Cases 1 through 3 illustrate the effect of varying damping. As implied above, the number of turns which the damper is inserted determines the relative amount of damping. Case 3 should be attempted in two configurations, first with the damper disconnected, then with it connected, but the plug removed. In the former configuration, there is clearly no damping, in the latter, there is negligible damping and added mass since the internal parts of the damper will move with the system. This demonstrates that fact that real dampers add mass to the system, so the best way to vary the damping without varying the system mass is to adjust the plug. The system damping should be relatively unchanged by connecting the damper and removing the plug, however, connecting the damper provides a significant increase in system mass which should be obvious from comparing the frequency of oscillations between the two cases.

Results

Figures 3 and 6-8 show typical experimental step responses compared to the identified linear models. Table 2 shows the identified parameters. The stiffness common to all four cases is 216.2 N/m. Since Eq. 10 contained more equations than unknowns, the identified parameters do not satisfy these equations exactly. That is to say, in particular, the differences between identified mass parameters may vary from the constraints of Eq. 9. This fact also accounts for the differences in identified damping constants. The damping was not changed at all between experiments, but the scatter in identified masses ripples through to the identified damping constants. The mass values shown in Table 1 account for the cart, rack, pinion, damper, and motor inertias. Thus, the numbers in Table 1 are expected to vary significantly from the specified added masses in cases 4-7. The static gain values are found by simply dividing the steady state response value by the step amplitude which was 0.5 volt for all cases.

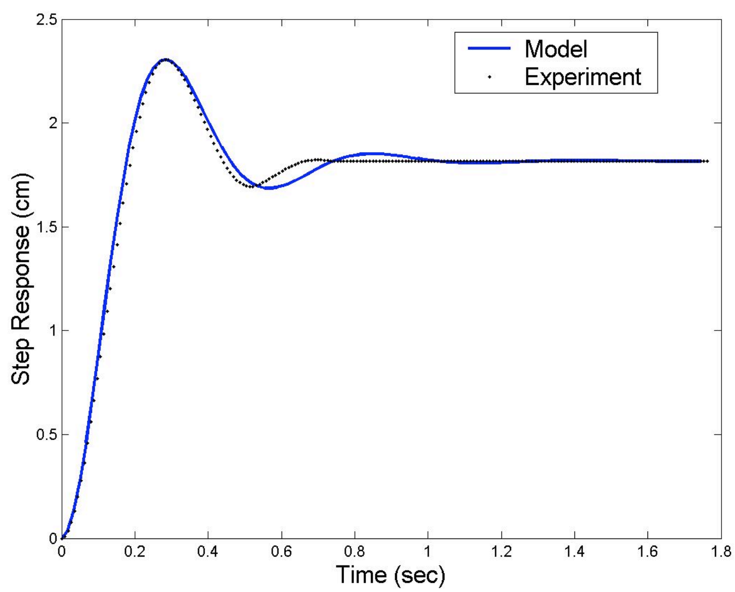

Fig 6. Comparison of Experimental and Parametric Model Step Response *m* = 1.4940kg, *k* = 216.2 N/m, *c* = 13.85 N-s/m

For each theoretical response, the identified parameters, *m*, *c,* and *k* were applied to the ODE model of Eq. 2. the system transfer function is then determined by a LaPlace transform of Eq. 2.

$$G(s) = \frac{K}{\frac{m}{k}s^2 + \frac{c}{k}s + 1} \tag{11}$$

Table 2. Identified Parameters

Case	Mass (kg)	Damping (N-s/m)	Static Gain *K*
4	0.9538	14.01	3.618
5	1.4940	13.85	3.633
6	2.4357	15.42	3.698
7	2.9951	16.19	3.876

The identified system transfer function for case 4 (Figure 3) is

$$G_4(s) = \frac{3.618}{0.004411s^2 + 0.06478s + 1} \tag{12}$$

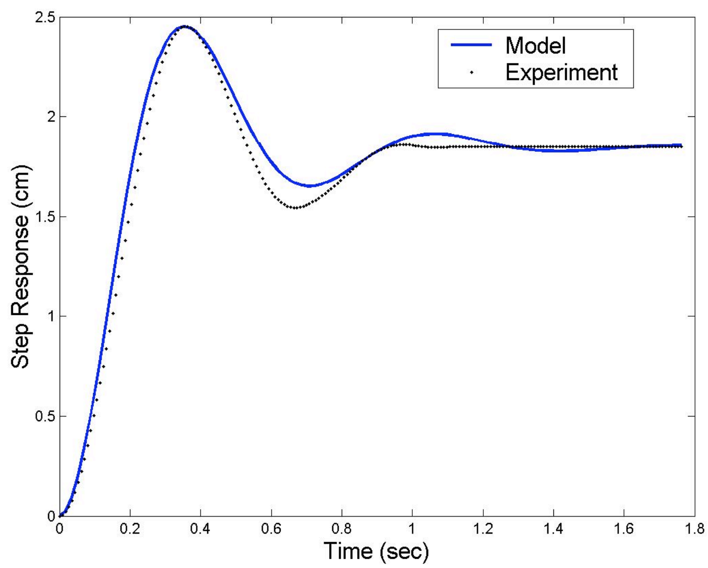

Fig. 7. Comparison of Experimental and Parametric Model Step Response *m* = 2.4357kg, *k* = 216.2 N/m, *c* = 15.42 N-s/m

The comparison plot in Figure 5 was made by obtaining the response of $G_4(s)$ to a step of magnitude 0.5.

The identified transfer function for case 5 (Figure 6) is

$$G_5(s) = \frac{3.633}{0.006909s^2 + 0.06407s + 1} \tag{13}$$

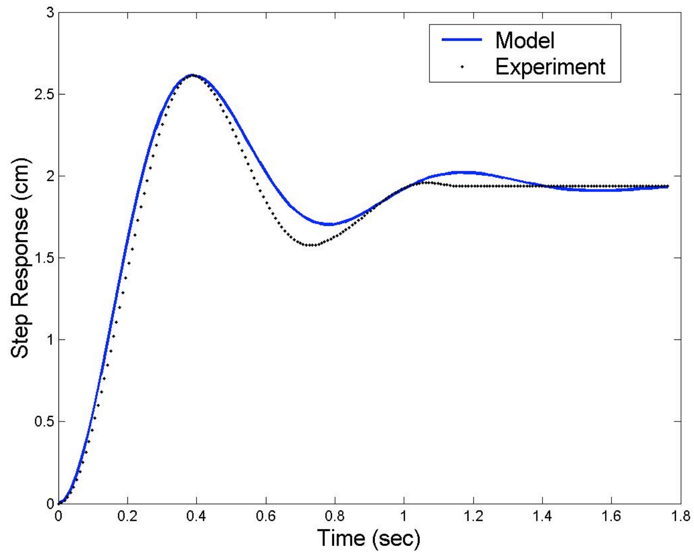

Fig. 8. Comparison of Experimental and Parametric Model Step Response *m* = 2.9951kg, *k* = 216.2 N/m, *c* = 16.19 N-s/m

For case 6 (Figure 7), the transfer function is

$$G_6(s) = \frac{3.698}{0.001126s^2 + 0.07131s + 1} \tag{14}$$

For case 7 (Figure 8), the transfer function is

$$G_7(s) = \frac{3.876}{0.001385s^2 + 0.07489s + 1} \tag{15}$$

By examining the experimental step responses shown, the student should realize that increasing the system mass decreases the damping ratio and system damped natural frequency. When stiffness and damping are held constant, increased mass causes the system to oscillate more slowly, and have a larger initial overshoot. All models match the corresponding experimental data through the first peak. There is a small amount of unmodeled Coulomb friction in the physical system which causes the experimental responses to settle out more quickly than the models. This effect is much more pronounced as the system mass is increased. This is shown by the progressively worse match in Figures 7 and 8.

Assessment

Seventeen of the students enrolled in the fall term 2004 filled out a confidential self-assessment of their abilities prior to and after taking the course. The following three questions are regarded as relevant to this work since only two of the physical labs involve detailed modeling of the spring-mass-damper system. The second lab, involving frequency response is described in detail in Burchett and Layton[2].

Table 3. Student Self-Assessment Questions.

Question	Strongly Agree	Agree	Disagree	Strongly Disagree	I Don't Know
As a result of this class, I understand the uses of models in ways that will help me in future classes.	35%	65%	0%	0%	0%
As a result of this class, I understand the limitations of models in ways that will help me in future classes.	35%	59%	6%	0%	0%
This class has helped me better understand how modeling systems can be applied in engineering situations.	35%	41%	18%	0%	6%

The students were also asked to rate their knowledge of various course topics and self-confidence in being able to apply the knowledge gained. They provided self-assessment scores both prior to and after taking the course. The following scales were provided to guide the students' self-assessment:

Knowledge: What you know regarding this concept area.

4 = High, I know the concept and I have applied it in this course.
3 = Moderate, I know the concept but I still have not applied it.
2 = Low, I have only heard about the concept, but do not know it well enough to apply it.
1 = No Clue, I do not know the concept.

Confidence: Level of confidence you have in your ability to solve problems in this area.

4 = High, I am confident that I understand and can apply the concept to problems.
3 = Moderate, I am somewhat confident that I understand the concept and I can apply it to a new problem.
2 = Low, I have heard of the concept but I am not sure that I can apply it.
1 = No Clue, I am not confident that I can apply the concept.

The Pre and post course knowledge and confidence scores for the most relevant concept areas are shown in Table 4. In every category there is approximately 0.7 or better improvement. This clearly shows that the students' assessment of their abilities was positively increased through lab experiences like the one described in this paper.

Concluding Remarks

This paper has shown a method for introducing sophomore engineering students to the use of step response as a modeling tool. In particular, this lab requires the student to determine the non-ideal parametric model characteristics that best fit a family of step responses. Along the way, students are see the effect of varying mass, stiffness and damping in a rectilinear vibrational system. A post-course survey showed that the students' knowledge and confidence in the topic of model determination were significantly increased.

Table 4. Student Self-Assessed Knowledge and Confidence.

Concept	Pre Course Knowledge	Pre Course Confidence	Post Course Knowledge	Post Course Confidence
Distinctions between a model and a real dynamical system	3.118	2.824`	3.705	3.588
Various approaches to modeling dynamical systems	2.705	2.471	3.471	3.235
Comparisons between predicted response of a mathematical model and the response of a physical system.	3.176	2.688	3.765	3.471

Acknowledgements

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Author Biography

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