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# ECE-205

# Exam 1

# Spring-2011

**Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.**

**You must show your work to receive credit.**

| | |
|---|---|
| **Problem 1** | ________/20 |
| **Problem 2** | ________/30 |
| **Problem 3** | ________/20 |
| **Problem 4** | ________/10 |
| **Problem 5** | ________/10 |
| **Problem 6** | ________/10 |

**Total** ____________

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**1) (20 points)** Assume we have a first order system with the governing differential equation

$$0.1\dot{y}(t) + y(t) = 2x(t)$$

The system has the initial value of 0, so $y(0) = 0$. The input to this system is

$$x(t) = \begin{cases} 0 & t < 0 \\ -1 & 0 \le t < 0.2 \\ -2 & 0.2 \le t < 0.5 \\ 3 & 0.5 < t \end{cases}$$

Determine the output of the system in each of the above time intervals. *Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!*

$y(t) = [y(t_0) - y(\infty)]e^{-(t-t_0)/\tau} + y(\infty) \qquad K=2 \quad \tau = 0.1$

① $A=-1 \quad KA = y(\infty) = -2 \quad t_0 = 0 \quad y(t_0) = 0$

$y(t) = [0-(-2)]e^{-t/0.1} + (-2) = \boxed{2e^{-t/0.1} - 2 = y(t) \quad 0 \le t \le 0.2}$

$y(0.2) = 2e^{-0.2/0.1} - 2 = -1.7293$

② $A=-2 \quad KA = y(\infty) = -4 \quad t_0 = 0.2 \quad y(t_0) = -1.7293$

$y(t) = [-1.7293-(-4)]e^{-(t-0.2)/0.1} + (-4) = \boxed{2.2707e^{-(t-0.2)/0.1} - 4 = y(t) \quad 0.2 \le t \le 0.5}$

$y(0.5) = 2.2707e^{-0.3/0.1} - 4 = -3.8869$

③ $A=3 \quad KA = y(\infty) = 6 \quad t_0 = 0.5 \quad y(t_0) = -3.8869$

$y(t) = [-3.8869 - 6]e^{-(t-0.5)/0.1} + 6 = \boxed{-9.8869e^{-(t-0.5)/0.1} + 6 = y(t) \quad t \ge 5}$

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**2) (30 points)** For the following three differential equations, assume the input is $x(t) = 4u(t)$ (the input is equal to four for time greater than zero), and the initial conditions are $y(0) = \dot{y}(0) = 0$

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit.

**a)** $\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2x(t)$

$4y_f = 2 \cdot 4 \quad y_f = 2 \quad (r+1)(r+4) = 0$

$y(t) = c_1 e^{-t} + c_2 e^{-4t} + 2$

$y(0) = c_1 + c_2 + 2 = 0 \qquad \dot{y}(t) = -c_1 e^{-t} - 4c_2 e^{-4t}$

$\dot{y}(0) = -c_1 - 4c_2 = 0$

adding $\Rightarrow$ $-3c_2 + 2 = 0 \quad \boxed{c_2 = 2/3} \quad c_1 = -4c_2 = \boxed{-8/3 = c_1}$

$$\boxed{y(t) = -\frac{8}{3}e^{-t} + \frac{2}{3}e^{-4t} + 2}$$

**b)** $\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 9x(t)$

$9y_f = 9(4) \quad y_f = 4 \quad (r+3)^2 = 0$

$y(t) = c_1 e^{-3t} + c_2 t e^{-3t} + 4$

$y(0) = c_1 + 4 = 0 \quad \boxed{c_1 = -4}$

$\dot{y}(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3t c_2 e^{-3t}$

$\dot{y}(0) = 0 = -3c_1 + c_2 \quad c_2 = 3c_1 = \boxed{-12 = c_2}$

$$\boxed{y(t) = -4e^{-3t} - 12te^{-3t} + 4}$$

**c)** $\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 6.5x(t)$

$13y_f = (6.5)(4) \quad y_f = 2 \quad r^2 + 4r + 13 = 0$

$r = \dfrac{-4 \pm \sqrt{16-52}}{2} = -2 \pm j3$

$y(t) = ce^{-2t}\sin(3t+\theta) + 2 \qquad y(0) = 0 = c\sin(\theta) + 2 \qquad c = -2/\sin(\theta)$

$\dot{y}(t) = -2ce^{-2t}\sin(3t+\theta) + 3ce^{-2t}\cos(3t+\theta)$

$\dot{y}(0) = 0 = -2\sin(\theta) + 3\cos(\theta)$

$\tan(\theta) = \frac{3}{2} \Rightarrow \theta = 56.3^\circ \Rightarrow c = -2/\sin(56.3^\circ) = -2.4037$

$$\boxed{y(t) = 2 - 2.4037e^{-2t}\sin(3t + 56.3^\circ)}$$

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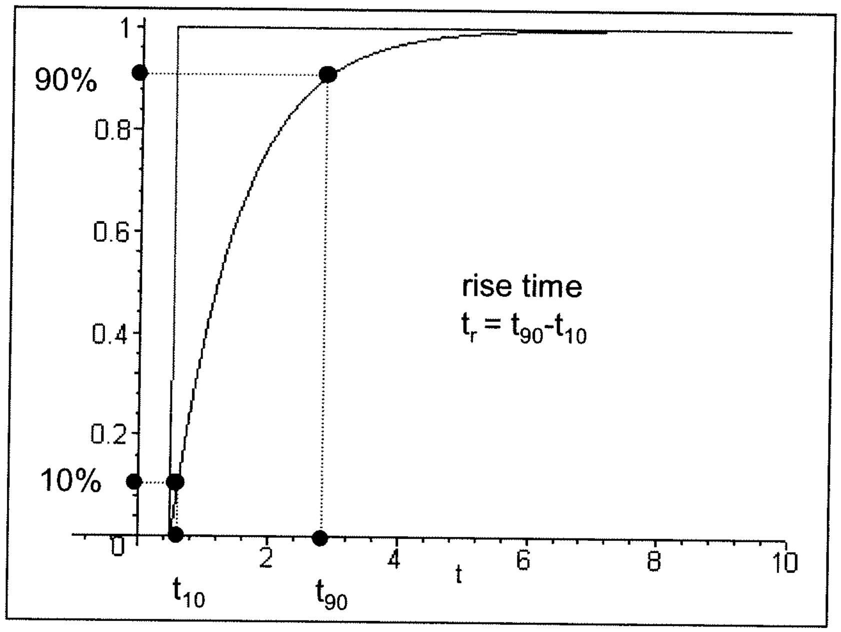
**3) (20 points)** For a simple series RC circuit the response of the system when the input is a unit step is

$$y(t) = 1 - e^{-t/RC} = 1 - e^{-t/\tau}$$

The 10-90% rise time, $t_r$, as shown below. The rise time is simply the amount of time necessary for the output to rise from 10% to 90% of its final value. Show that for this system the rise time is given by $t_r = \tau \ln(9)$




$y(t_{10}) = 0.1 = 1 - e^{-t_{10}/\tau}$

$e^{-t_{10}/\tau} = 0.9$

$y(t_{90}) = 0.9 = 1 - e^{-t_{90}/\tau}$

$e^{-t_{90}/\tau} = 0.1$

$$\frac{e^{-t_{10}/\tau}}{e^{-t_{90}/\tau}} = e^{(t_{90}-t_{10})/\tau} = e^{t_r/\tau} = 9$$

$$\frac{t_r}{\tau} = \ln 9$$

$$\boxed{t_r = \tau \ln 9}$$

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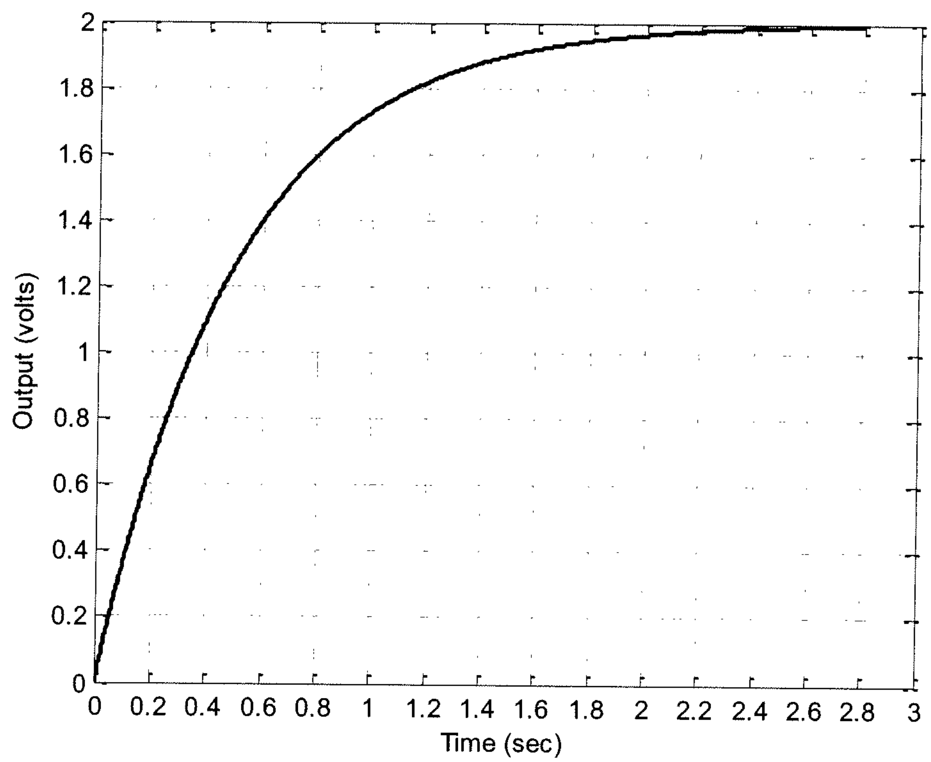
**4) (10 points)** The initial portion of the response of a first order circuit to a step input is shown in the figure below. The steady state value of the output is 2.0 volts. Using this data, estimate the time constant of the system. ***Show your work or you receive no credit!*** *(This requires very little work, but be sure to show it.)*




$0.98y(\infty) = (0.98)(2) = 1.96$ this happens at $T_s = 4\tau \approx 2$

$\tau \approx \frac{2}{4} = \boxed{0.5 \text{ sec} = \tau}$

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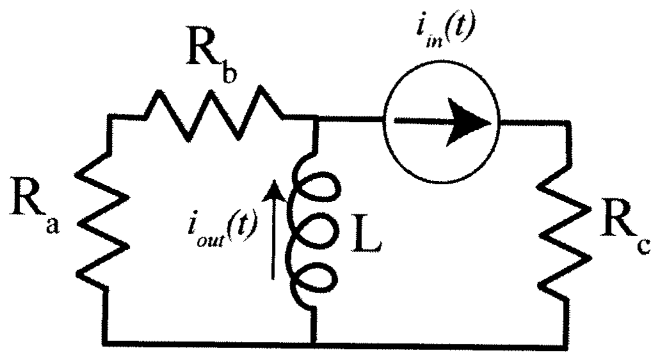
**5) (10 points)** For the following circuit, determine an expression for the time constant and the static gain. *(This can pretty much be done by inspection.)*




$$\boxed{\begin{aligned} R_{th} &= R_a + R_b \\ \tau &= L/R_{th} \\ K &= 1 \end{aligned}}$$

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**6) (10 points)** Using integrating factors, solve the following differential equation for an arbitrary input $x(t)$ for the initial conditions $t_0 = 0$ and $y(0) = 1$. Note that your solution will involve an integral, since you do not know what the input $x(t)$ is. ***Show your work or you receive no credit!***

$$\frac{1}{2}\dot{y}(t) - ty(t) = 3x(t)$$

$$\dot{y}(t) - 2ty(t) = 6x(t)$$

$$\frac{d}{dt}\left[y(t)e^{-t^2}\right] = 6e^{-t^2}x(t)$$

$$y(t)e^{-t^2} - y(t_0)e^{-t_0^2} = \int_{t_0}^{t} 6e^{-\lambda^2}x(\lambda)d\lambda$$

$$y(t)e^{-t^2} = 1 + 6\int_0^t e^{-\lambda^2}x(\lambda)d\lambda$$

$$\boxed{y(t) = e^{t^2} + 6e^{t^2}\int_0^t e^{-\lambda^2}x(\lambda)d\lambda}$$