ECE 300
Signals and Systems
Homework 6

Due Date: Tuesday January 23 at 12:40 PM **Exam 2, Thursday January 25**

Problems:

1. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal lowpass filter that eliminates all signals with frequency content higher than 1.25 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

c) Determine the average power in $y(t)$.

d) Plot the spectrum (magnitude and phase) for $x(t)$. Include the DC through second harmonic. Accurately label your plot.

2. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k a_k e^{jkt}$$

where

$$a_k = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no e^{ja} terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

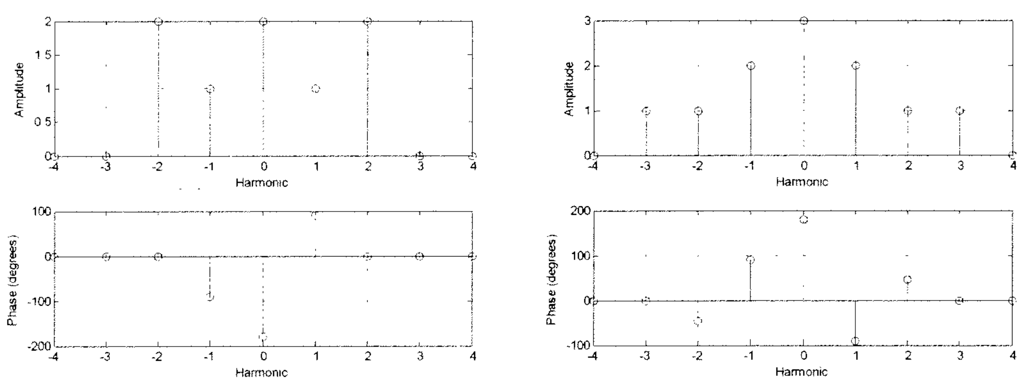
3. K & H, Problem 5.1. Use the example we did in class to get the Fourier series coefficients for part c.

4. K & H, Problem 5.3.

5. K & H, Problem 5.12. Note that $y(t) = x(t) - x(t-1)$. You need to write c_k^y in terms of c_k^x.

6. K & H, Problme 5.13.

7. The output of a LTI system, $y(t)$, has the following spectrum shown on the left, while the system transfer function, $H(k\omega_o)$, has the spectrum shown on the right. Assume all angles are multiples of 45 degrees.

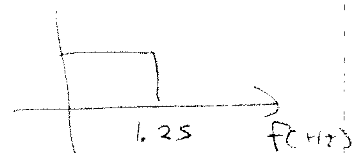


a) Determine (sketch) the spectrum (magnitude and phase) of the input to the system, $x(t)$.

b) If $x(t)$ has the fundamental period $T = 2$ seconds, determine an analytical expression for $x(t)$ in terms of sine, cosines, and constants.

Problem Set # 6 ECE-300 ①

#1 $x(t) = e^{-t} \quad 0 < t < 2 \qquad x(t) = \sum_K \frac{0.4323}{1 + jK\pi} e^{jK\pi t}$



(a) $P_{ave} = \frac{1}{2}\int_0^2 e^{-2t}\,dt$

$= \frac{1}{2} \left. \frac{e^{-2t}}{-2} \right|_0^2 = \frac{1}{4}(1 - e^{-4}) = \boxed{0.24542 = P_{ave}}$

(b)

K	Kf_0
0	0
1	1/2
2	1
3	3/2
4	2

$c_0 = 0.4323$

$c_1 = \frac{0.4323}{1 + j\pi} = 0.1311 \angle -72.34°$

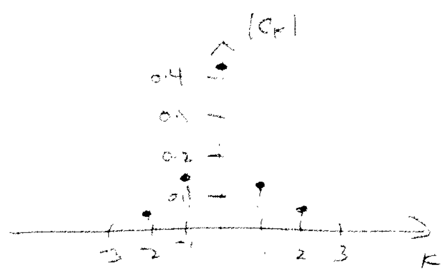
$c_2 = \frac{0.4323}{1 + j2\pi} = 0.0680 \angle -80.96°$

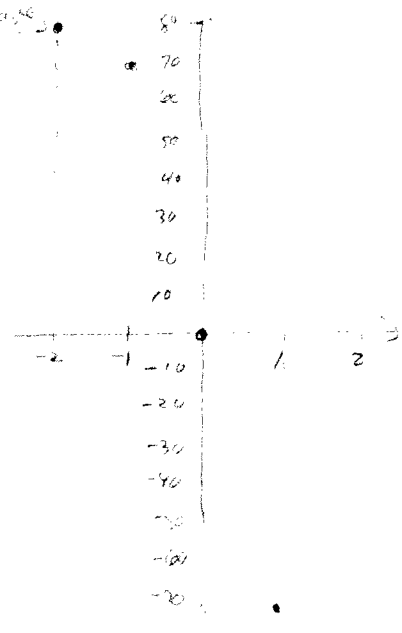
$y(t) = c_0 + 2|c_1|\cos(\omega_0 t + \angle c_1) + 2|c_2|\cos(2\omega_0 t + \angle c_2)$

$y(t) = 0.4323 + 0.2622\cos(\pi t - 72.34°) + 0.1360\cos(2\pi t - 80.96°)$

(c) $P^y_{ave} = |c_0|^2 + 2|c_1|^2 + 2|c_2|^2 = \boxed{0.2305 = P^y_{ave}}$

(d)

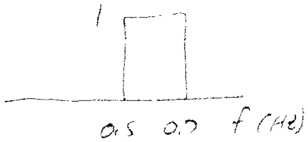




Problem Set #6 ECE 306

#2 $x(t) = t^2 \quad -\pi \le t \le \pi \qquad x(t) = \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{2(-1)^k}{k^2} e^{jkt}$

a)



Remove all signals with frequencies outside 0.5 to 0.7 Hz

$f_0 = \frac{1}{T_0} = \frac{1}{2\pi} = 0.1591 5$

K	Kf_0
0	0
1	0.15915
2	0.31831
3	0.47746
4	0.63662
5	0.79577

$c_4 = \frac{2(-1)^4}{4^2} = 0.125 \angle 0°$

$y(t) = 2|c_4| \cos(4\omega_0 t + \angle c_4) = 2(0.125)\cos(4t) = \boxed{0.25\cos(4t) = y(t)}$

$P_{ave}^x = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 dt = \frac{1}{2\pi}\frac{t^5}{5}\Big|_{-\pi}^{\pi} = \frac{1}{10\pi} 2\pi^5 = \frac{\pi^4}{5} = \boxed{19.4818 = P_{ave}^x}$

$P_{ave}^y = 2|c_4|^2 = 0.03125 \qquad \frac{P_{ave}^x}{P_{ave}^y} = \frac{0.03125}{19.48} = 0.0016 = \boxed{0.16\% = \frac{P_{ave}^x}{P_{ave}^y}}$

b) $y(t) = t^2 - 0.25\cos(4t)$

$P_{ave}^y = P_{ave}^x - 0.03125 = 19.45057 \qquad \frac{P_{ave}^y}{P_{ave}^x} = \frac{19.45057}{19.4818} = 0.9984$

$= 99.84\% = \frac{P_{ave}^x}{P_{ave}^y}$

#3

5.1

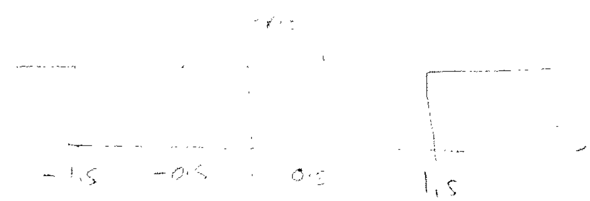
$$H(\omega) = \begin{cases} 1 & 2 \le |\omega| \le 7 \\ 0 & \text{all others} \end{cases}$$

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

a) $x(t) = 2 + 3\cos(3t) - 5\sin(6t - 30^\circ) + 4\cos(13t - 20^\circ)$

b) $x(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k}\cos(2kt)$

c) $x(t)$ shown below



a) $y(t) = 2H(0) + 3|H(3)|\cos(3t + \angle H(3)) - 5|H(6)|\sin(6t - 30^\circ + \angle H(6))$

$\quad + 4|H(13)|\cos(13t - 20^\circ - \angle H(13))$

$$\boxed{y(t) = 3\cos(3t) - 5\sin(6t - 30^\circ)}$$

b) $y(t) = 1H(0) + \sum_{k=1}^{\infty} \frac{1}{k}|H(2k)|\cos(2kt + \angle H(2k))$

$$\boxed{y(t) = \cos(2t) + \frac{1}{2}\cos(4t) + \frac{1}{3}\cos(6t)}$$

c) $C_k^x = \frac{A\tau}{T_0}\operatorname{sinc}\left(k\frac{\tau}{T_0}\right) = \frac{(1)(1)}{2}\operatorname{sinc}\left(\frac{k1}{2}\right) = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right)$

$T_0 = 2 \qquad \omega_0 = \frac{2\pi}{T_0} = \pi$

$x(t) = \sum_k \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right)e^{jk\pi t}$

For $k=0$ $\omega=0$
$k=1$ $\omega=\pi$
$k=2$ $\omega=2\pi$ } only terms
$k=3$ $\omega=3\pi$

$C_1^y = C_1^x H(\pi) = \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}\right) = \frac{1}{\pi}$

$C_2^y = C_2^x H(2\pi) = \frac{1}{2}\operatorname{sinc}\left(\frac{2}{2}\right) = 0$

$y(t) = 2|C_1^y|\cos(\omega_0 t + \angle C_1^y) = \boxed{\frac{2}{\pi}\cos(\pi t) = y(t)}$

Problem set #6 | ECE-300

#4 (5.3) $H(\omega) = \dfrac{1}{j\omega + 1}$ $\quad x \rightarrow \boxed{H} \rightarrow y$

a) $x(t) = \cos(t)$

b) $x(t) = \cos(t + 45^\circ)$

$\omega_0 = 1 \qquad H(j\omega_0) = \dfrac{1}{1+j} = \dfrac{1}{\sqrt{2}} \measuredangle -45^\circ$

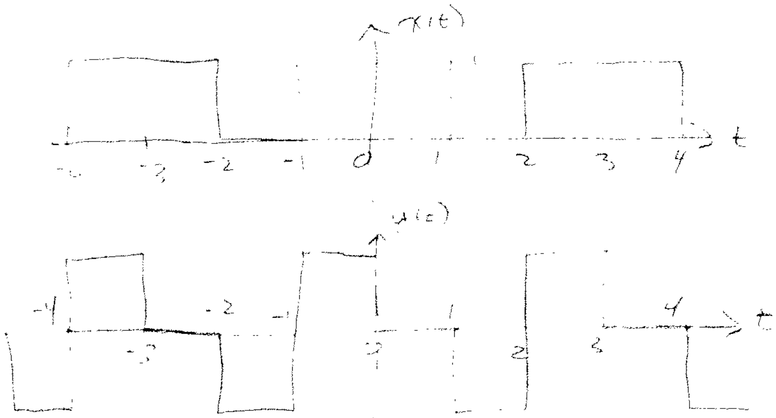
a) $y(t) = |H(j\omega_0)| \cos\left(\omega_0 t + \measuredangle H(j\omega_0)\right) = \boxed{\dfrac{1}{\sqrt{2}} \cos(t - 45^\circ) = y(t)}$

b) $y(t) = |H(j\omega_0)| \cos(\omega_0 t + \measuredangle H(j\omega_0) + 45^\circ) = \boxed{\dfrac{1}{\sqrt{2}} \cos(t) = y(t)}$

#5 (5.12) $H(\omega) = b - a e^{j\omega c}$ $\quad -\infty < \omega < \infty$

a, b, c are real numbers

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$



$y(t) = x(t) - x(t-1)$

$$C_k^x = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k^y = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} \left[x(t) - x(t-1)\right] e^{-jk\omega_0 t}\, dt$$

$$= \underbrace{\frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt}_{C_k^x} - \frac{1}{T_0}\int_{T_0} x(t-1)\, e^{-jk\omega_0 t}\, dt$$

let $\lambda = t-1$ $\quad \lambda + 1 = t$

$d\lambda = dt$

limits don't change since still integrating over one period

$$-\frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-jk\omega_0(\lambda+1)}\, d\lambda$$

$$= -e^{-jk\omega_0}\left[\frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-jk\omega_0\lambda}\, d\lambda\right]$$

$$= -e^{-jk\omega_0} C_k^x$$

So $C_k^y = C_k^x - e^{-jk\omega_0} C_k^x = \left(1 - e^{-jk\omega_0}\right) C_k^x$

but $C_k^y = C_k^x H(k\omega_0) \Rightarrow H(k\omega_0) = 1 - e^{-jk\omega_0}$

$\Rightarrow \boxed{a = b = 1 \quad c = -1}$

Problem #6 [illegible]

#6 (5.13) $x(t) = [illegible]\,4\cos(2\pi t) + 8\sin(3\pi t - 90^\circ)$

$y(t) = 2 - 2\sin(2\pi t)$ $x \rightarrow \boxed{H} \rightarrow y$

what is H?

(b) $y(t) = 2 - 2\sin(2\pi t) = 1\,H(0) + 4\,|H(2\pi)|\cos(2\pi t + \measuredangle H(2\pi))$

$+ 8\,|H(3\pi)|\sin(3\pi t - 90^\circ + \measuredangle H(3\pi))$

$H(0) = 2$ $H(3\pi) = 0$

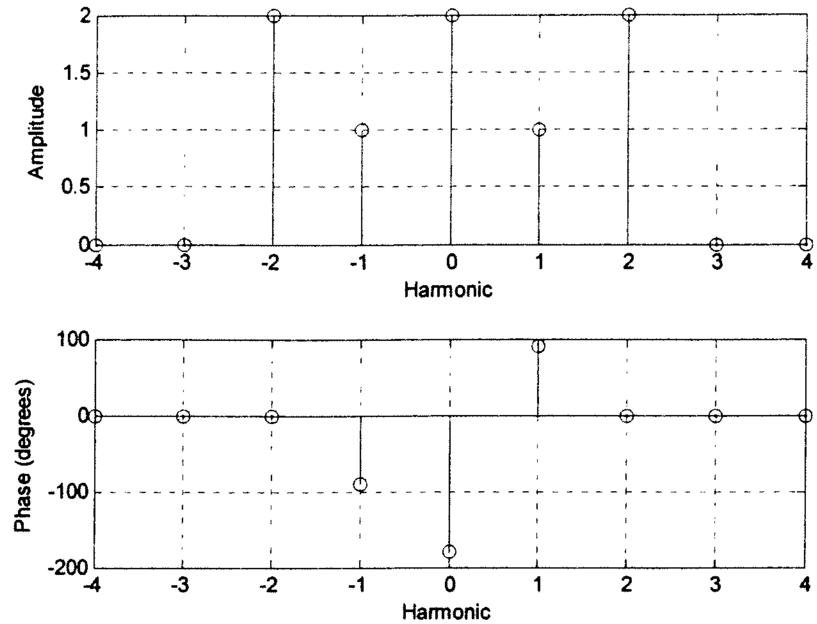
$|H(2\pi)| = \frac{1}{2}$ $\measuredangle H(2\pi) = +90^\circ$

$\boxed{H(0) = 2 \quad H(2\pi) = \frac{1}{2}e^{j\frac{\pi}{2}} \quad H(3\pi) = 0}$

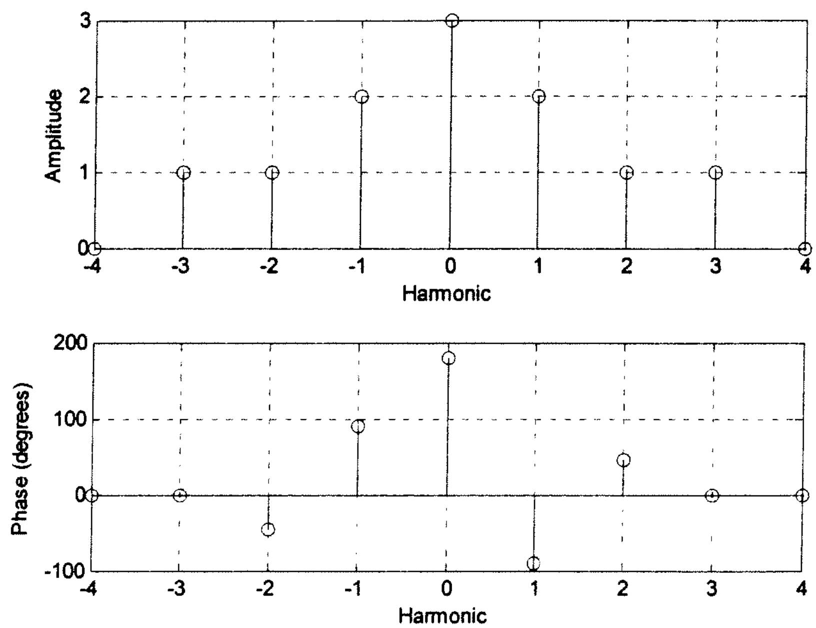
(a) we can only determine $H(\omega)$ at $\omega = 0$, $\omega = 2\pi$, $\omega = 3\pi$

Problem set #6 ECE-300

(A7) 8. The output of a LTI system, $y(t)$, has the following spectrum:



The system transfer function, $H(k\omega_o)$, has the following spectrum:



$c_k^y = c_k^x H(jk\omega_0)$ $\therefore$ $c_k^x = \dfrac{c_k^y}{H(jk\omega_0)} = \dfrac{|c_k^y| \measuredangle c_k^y}{|H(jk\omega_0)| \measuredangle H(jk\omega_0)}$

$|c_k^x| = \dfrac{|c_k^y|}{|H(jk\omega_0)|}$ $\quad \measuredangle c_k^x = \measuredangle c_k^y - \measuredangle H(jk\omega_0)$

$c_0^x = \frac{2}{3}$ $\quad \measuredangle c_0^x = (-180^\circ) - (180^\circ) = -360^\circ = 0^\circ$

$|c_1^x| = \frac{1}{2}$ $\quad \measuredangle c_1^x = 90^\circ - (-90^\circ) = 180^\circ$

(continued)

Problem Set 6 ECE-300

(#7) (continued)

$$|c_2^x| = \frac{2}{1} \qquad \measuredangle c_2^x = (0) - (90^\circ) = -45^\circ$$

$$|c_3^x| = \frac{0}{1} = 0 \qquad \measuredangle c_3^x = (0) - (0) - 0^\circ$$

so $c_0^x = \frac{2}{3} \measuredangle 0^\circ$

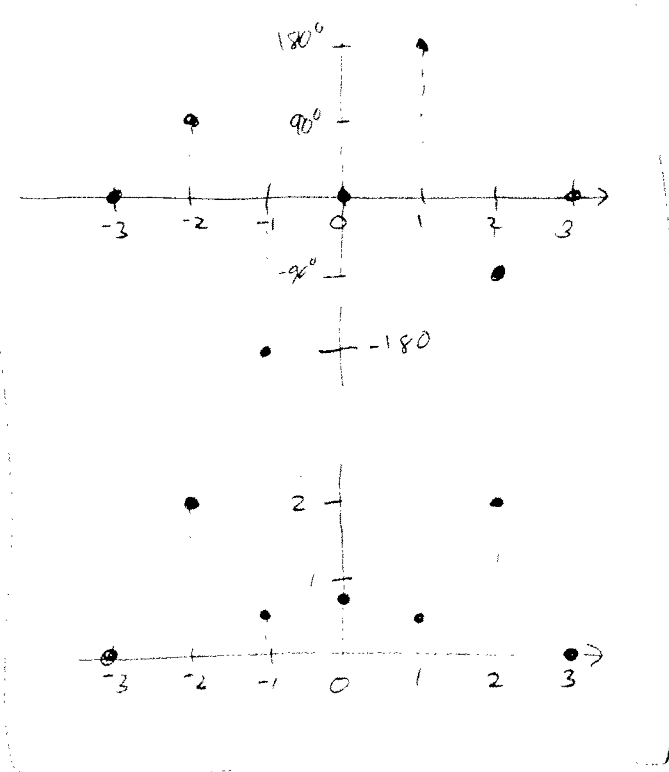
$c_1^x = \frac{1}{2} \measuredangle 180^\circ$

$c_2^x = 2 \measuredangle -90^\circ$

$c_k^x = 0 \qquad k \geq 3$

Use the relationships $|c_k^x| = |c_{-k}^x|$

$\measuredangle c_{-k}^x = -\measuredangle c_k^x$



$$x(t) = c_0^x + 2|c_1^x|\cos(\omega_0 t + \measuredangle c_1^x) + 2|c_2^x|\cos(2\omega_0 t + \measuredangle c_2^x)$$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$x(t) = \frac{2}{3} + \cos(\pi t + 180^\circ) + 4\cos(2\pi t - 45^\circ)$$

$$x(t) = \frac{2}{3} - \cos(\pi t) + 4\cos(2\pi t - 45^\circ)$$