Lecture 6

Magnetic Fields
Electromotive Force (EMF)
Magnetomotive Force (MMF)
Maxwell's Equations in Integral Form

Magnetic Force Field

- Just as the electric field **E** was defined to "explain" (patch over) the force acting between two stationary charges, the magnetic field **B** was defined to "explain" the force acting between two current-carrying loops of wire:
- $\mathbf{F} = Q\mathbf{V} \times \mathbf{B}$
- $\mathbf{dF} = I\mathbf{dL} \times \mathbf{B}$

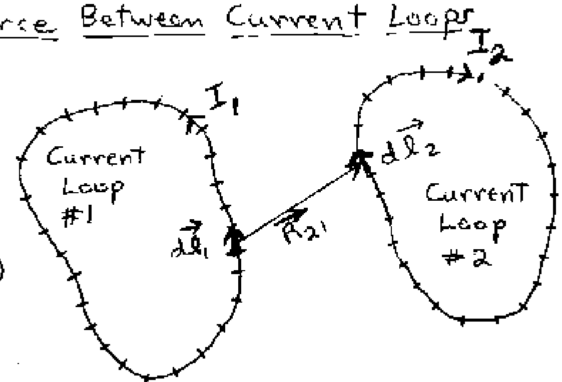
1-8

C. Magnetic Field

1. Ampere's Law of Force Between Current Loops

$$\vec{F}_2 = k \oint_{C_1} \oint_{C_2} \frac{I_2 \, d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{R}_{12})}{R_{12}^3}$$

$$\vec{F}_1 = k \oint_{C_1} \oint_{C_2} \frac{I_1 \, d\vec{l}_1 \times (I_2 d\vec{l}_2 \times \vec{R}_{21})}{R_{21}^3}$$



where

$$k = \frac{\mu}{4\pi} = \frac{(\mu_0 \mu_R)}{4\pi}$$

and
μ_0 = permeability of free space = $4\pi \times 10^{-7}$ H/m
μ_R = relative permeability of medium (= 1 for free space)

Force between current elements:

$$d\vec{F}_2 = \frac{\mu}{4\pi} I_2 \, d\vec{l}_2 \times \frac{I_1 \, d\vec{l}_1 \times \vec{R}_{12}}{R_{12}^3} = I_2 \, d\vec{l}_2 \times \underbrace{\left(\frac{\mu}{4\pi} \frac{I_1 \, d\vec{l}_1 \times \vec{R}_{12}}{R_{12}^3} \right)}_{d\vec{B}_2}$$

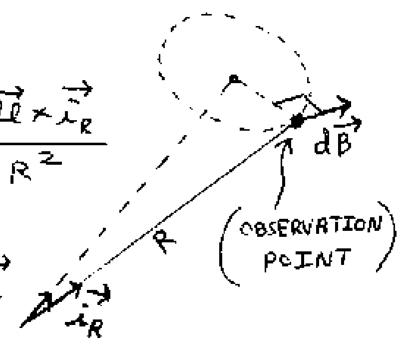
Likewise,

$$d\vec{F}_1 = I_1 \, d\vec{l}_1 \times \underbrace{\left(\frac{\mu}{4\pi} \frac{I_2 \, d\vec{l}_2 \times \vec{R}_{21}}{R_{21}^3} \right)}_{d\vec{B}_1}$$

In general we have

$$d\vec{B} = \frac{\mu}{4\pi}\,\frac{I\,d\vec{l} \times \vec{R}}{R^3} = \frac{\mu}{4\pi}\,\frac{I\,d\vec{l} \times \vec{i}_R}{R^2}$$

(BIOT- SAVART LAW)

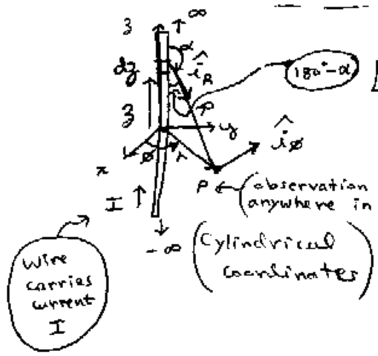


$\vec{B}$ is called the "magnetic flux density" with units of $\frac{\text{Newtons}}{(\text{Coul/Sec})(\text{meter})}$

Therefore the force on a current-carrying wire segment $I_1 d\mathbf{l_1}$ in the presence of the magnetic flux density d**B** produced by a current-carrying wire segment Id**l** is given by:

$$d\mathbf{F} = I_1 d\mathbf{l_1} \text{ x } d\mathbf{B}$$

Example #1: How Biot-Savart Law may be applied to determine $\vec{B}$ due to Infinitely long, Straight Wire (1-8A)

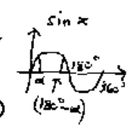


(180°−α)

$$[d\vec{B}]_{(r,\phi,0)} = \frac{\mu_0}{4\pi}\,\frac{I\,dz\,\hat{i}_z \times \hat{i}_R}{R^2}$$

$$= \frac{\mu_0 I\,dz\,\sin\alpha}{4\pi R^2}\,\hat{i}_\phi$$

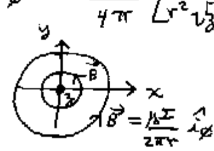
$$= \frac{\mu_0 I\,dz}{4\pi R^2}\left[\underbrace{\frac{r}{R}}_{\sin(180°-\alpha)\,=\,\sin\alpha}\right]\hat{i}_\phi$$



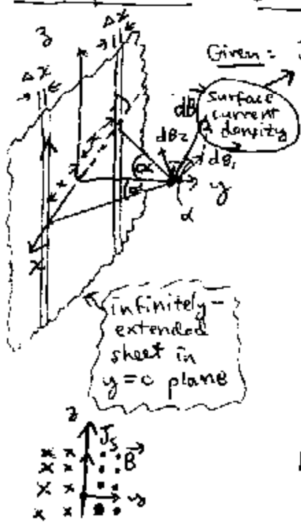
$$[\vec{B}]_{r,\phi,0} = \int_{z=-\infty}^{\infty} d\vec{B} = \int_{z=-\infty}^{\infty} \frac{\mu_0 I r}{4\pi R^3}\,dz\,\hat{i}_\phi$$

$$= \int_{z=-\infty}^{\infty} \frac{\mu_0 I r}{4\pi\left(\sqrt{r^2+z^2}\right)^3}\,dz\,\hat{i}_\phi = \frac{\mu_0 I r}{4\pi}\left[\frac{z}{r^2\sqrt{z^2+r^2}}\right]_{z=-\infty}^{\infty}$$

$$\boxed{[\vec{B}]_{r,\phi,0} = \frac{\mu_0 I}{2\pi r}\,\hat{i}_\phi}$$



Example #2 $\vec{B}$ due to infinite, plane sheet of current



Given: $J_s = J_{s0}\,\hat{i}_z$ A/m (uniformly distributed current density)

From symmetry, we see that

B = $B_x(y)$ **i**$_x$ (from result of example #1) "I"

$$\therefore\ d\vec{B} = -2dB_1(\cos\alpha)\hat{i}_x = -\frac{\mu_0 (J_{s0}\,dx)}{\pi(x^2+y^2)^{1/2}} \underbrace{\left(\frac{y}{\sqrt{x^2+y^2}}\right)}_{\cos\alpha}$$

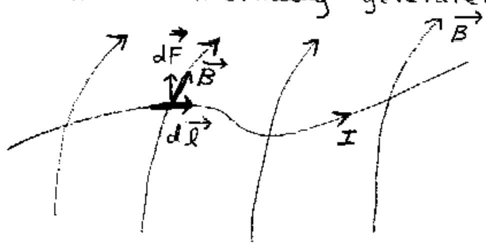
$$\vec{B} = \int_{x=0}^{\infty} d\vec{B} = -\int_{x=0}^{\infty} \frac{\mu_0 J_{s0}\,y}{\pi(x^2+y^2)}\,dx\,\hat{i}_x = -\frac{\mu_0 J_{s0}\,y}{\pi}\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right]_{x=0}^{\infty} \mathbf{i}_x$$

$$\boxed{\vec{B} = \mp\frac{\mu_0 J_{s0}}{2}\hat{i}_x \text{ for } y \gtrless 0} \quad \text{or} \quad \boxed{\vec{B} = \frac{\mu_0}{2}\vec{J}_s \times \hat{i}_n}$$

(1-9)

2. Lorentz Force Equation

Now consider a current-carrying wire that is immersed in an externally-generated $\vec{B}$ field.



Ampere's Law of Force predicts that the incremental force, $d\vec{F}$, exerted on an incremental segment of wire, $d\vec{\ell}$, is given by

$$d\vec{F} = I\,d\vec{\ell} \times \vec{B}$$

$$= \frac{dQ}{dt}\,d\vec{\ell} \times \vec{B}$$

$$= dQ\,\frac{d\vec{\ell}}{dt} \times \vec{B}$$

$$= dQ\ \vec{v} \times \vec{B}$$

WHERE DID THE "$d\vec{B}$" in Ampere's Force Law go to?

Vanishingly small Test Charge

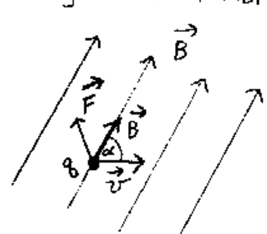
velocity of test charge

Thus, magnetic field ($\vec{B}$) may be defined in terms of the force on a vanishingly small moving test charge, just as electric field ($\vec{E}$) is defined in terms of the force on a stationary test charge!

Physical Interpretation of Magnetic Field (1-10)

If a test charge q moves with a velocity $\vec{v}$ and subsequently experiences a force $\vec{F}$ due to this movement (imagine that no force is exerted on this charge when it is at rest), then the region is inhabited by a magnetic field $\vec{B}$ such that

$$\boxed{\vec{F} = q\,\vec{v} \times \vec{B}}$$



where $|\vec{F}| = q v B$

direction of $\vec{F}$ = perpendicular to both $\vec{v}$ and $\vec{B}$. (directed in sense of right-hand rule)

We define $\vec{B}$ as

$$\vec{B} = \lim_{qv \to 0} \frac{\vec{F}_m \times \vec{i}_m}{q v}$$

where $\vec{F}_m$ is the maximum force experienced by the test charge and $\vec{i}_m$ is the unit vector along the direction of $\vec{v}$ for which this maximum force is observed. (Note that the angle between $\vec{F}_m$ and $\vec{i}_m$ will be 90°.)

Units of $\vec{B}$ $= \dfrac{\text{newtons}}{(\text{Coulomb})(\text{meter/sec})} = \dfrac{\text{newtons}}{\text{Coulomb}} \cdot \dfrac{\text{second}}{\text{meter}}$

$= \dfrac{\text{newton-meter}}{\text{coulomb}} \cdot \dfrac{\text{second}}{(\text{meter})^2} = \dfrac{\text{volt-second}}{(\text{meter})^2}$

= Weber/meter^2

(Weber = volt-second)

(1-11)

Just as $\vec{J}$ has units of Amperes/(meter)^2 and is called the "current density," $\vec{B}$ has units of webers/(meter)^2 and is called magnetic flux density.

The Lorentz Force Equation:

If both $\vec{E}$ and $\vec{B}$ exist in a region, then

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

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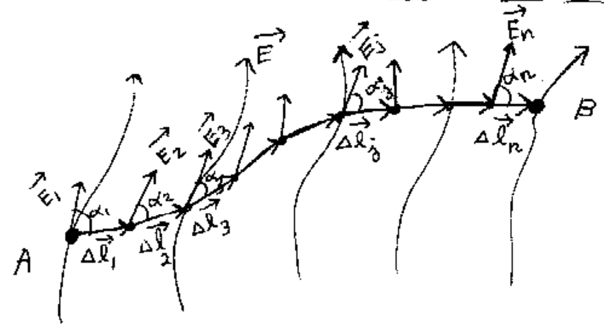
D. Electromotive Force (Voltage): The Line Integral



The voltage or Electromotive force (EMF) at B w.r.t. A is the work done in carrying a 1-coulomb charge from point A to point B:

$$V_{BA} \cong -E_1\cos\alpha_1\,\Delta l_1 - E_2\cos\alpha_2\,\Delta l_2 - \cdots - \underbrace{E_n\cos\alpha_n}_{\text{Force exerted along the direction of charge motion}}\,\underbrace{\Delta l_n}$$

$$= -\sum_{j=1}^{n} E_j\cos\alpha_j\,\Delta l_j$$

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$$= -\sum_{j=1}^{n} E_j\cos\alpha_j\,\Delta l_j$$

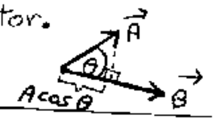
$$= -\sum_{j=1}^{n} \vec{E}_j \cdot \Delta\vec{l}_j$$

"dot product"

Remember:

The dot product of 2 vectors is the magnitude of one vector multiplied by the magnitude of the component of the other vector that lies along the direction of the first vector.

Thus $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} = AB\cos\theta$



1-18

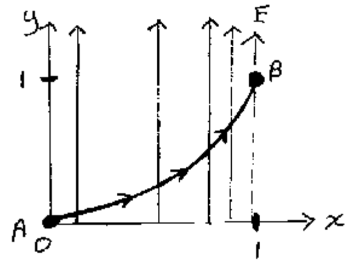
In the limit as $\Delta l \to 0$, and $n \to \infty$

$$V_{AB} = \lim_{n\to\infty} \sum_{j=1}^{n} \vec{E}_j \cdot \vec{\Delta l}_j = \int_A^B \vec{E}\cdot d\vec{l} =$$ "Line Integral of $\vec{E}$ over the specified path from A to B"

The subscript order has been switched to get rid of (−) sign!

<u>Example:</u> Consider $\vec{E} = x\,\vec{i}_y$ and the path $y = x^2$, $z = 0$

From the point A (0,0,0) to the point B (1,1,0)

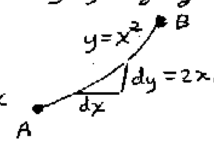


In general, $d\vec{l}$ is given by:

$$d\vec{l} = dx\,\vec{i}_x + dy\,\vec{i}_y + dz\,\vec{i}_z$$

<u>but</u> $y = x^2$

$$\Rightarrow \frac{dy}{dx} = 2x$$



<u>and</u> $z = 0 \Rightarrow dz = 0$

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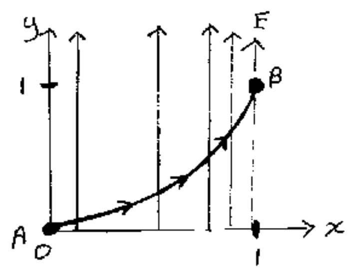

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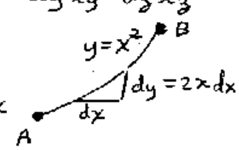


<u>and</u> $z = 0 \Rightarrow dz = 0$

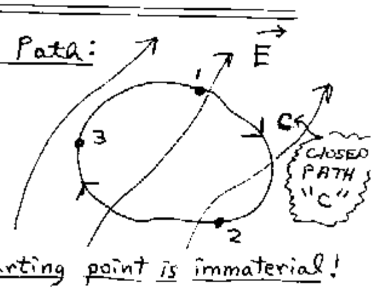
$$\Rightarrow d\vec{l} = dx\,\vec{i}_x + 2x\,dx\,\vec{i}_y$$

$$\therefore\ \vec{E}\cdot d\vec{l} = (x\,\vec{i}_y)\cdot(dx\,\vec{i}_x + 2x\,dx\,\vec{i}_y) = 2x^2\,dx$$

$$V_{AB} = \int_A^B \vec{E}\cdot d\vec{l} = \int_{(0,0,0)}^{(1,1,0)} \vec{E}\cdot d\vec{l} = \int_{x=0}^{x=1} 2x^2\,dx = \underline{\tfrac{2}{3}\ \text{Volt}}$$

<u>Line Integration Around a Closed Path:</u>

$$\oint_C \vec{E}\cdot d\vec{l} = \int_1^2 \vec{E}\cdot d\vec{l} + \int_2^3 \vec{E}\cdot d\vec{l} + \int_3^1 \vec{E}\cdot d\vec{l}$$



$\oint_C \vec{E}\cdot d\vec{l}$ is the work done per unit charge in moving a test charge around the closed path "C". <u>The starting point is immaterial!</u>

E. Magnetic Flux: Surface Integration

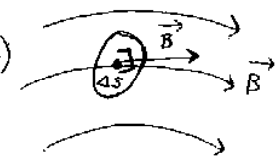
(1-19)

Recall that the magnetic flux density vector ($\vec{B}$) is measured in webers/m^2. Just as we envisioned the current density vector, $\vec{J}$, as the current/unit area flowing through a cross-sectional area (ΔS), oriented normal to the direction of $\vec{J}$ (to maximize the flow through ΔS), we can envision $\vec{B}$, the flux density vector as the magnetic flux (or field) per unit area "flowing" through a cross-sectional area ΔS oriented normal to the direction of "magnetic field flow" $\vec{B}$ (to maximize the flow through ΔS).

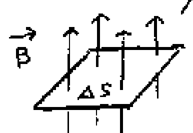
The magnetic flux crossing the cross section ΔS that is oriented normal to $\vec{B}$ is given by

$$\Delta \phi \cong |\vec{B}| \, \Delta S \quad \text{(webers)}$$

(This becomes more accurate as $\Delta S \to 0$ if $\vec{B}$ is not uniform.)

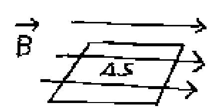


Thus, the net magnetic field (flux) crossing an arbitrarily oriented surface area "S" (in webers)



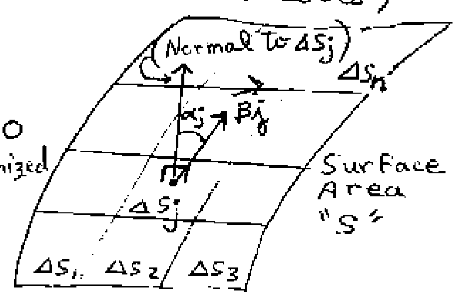
$\vec{B} \perp \Delta S \Rightarrow \Delta \phi = B \Delta S$

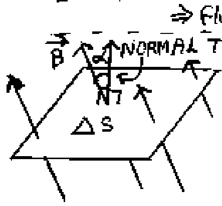
$\Rightarrow$ flux maximized



$\vec{B} // \Delta S \Rightarrow \Delta \phi = 0$

$\Rightarrow$ flux minimized





IN GENERAL:

$$\Delta \phi = \underbrace{B \cos\alpha}_{\text{COMPONENT OF } \vec{B} \text{ ALONG DIRECTION NORMAL TO } \Delta S} \Delta S$$

The net magnetic flux crossing the surface S (1-20)
must therefore be given (approximately) by:

$$[\phi]_S \cong \Delta\phi_1 + \Delta\phi_2 + \Delta\phi_3 + \cdots + \Delta\phi_n$$

$$= B_1 \cos\alpha_1 \Delta S_1 + B_2 \cos\alpha_2 \Delta S_2 + \cdots + B_n \cos\alpha_n \Delta S_n$$

$$= \sum_{j=1}^{n} B_j \cos\alpha_j \Delta S_j = \sum_{j=1}^{n} \vec{B}_j \cdot \Delta S_j \vec{i}_{nj}$$

unit vector normal to ΔS_j

$$= \sum_{j=1}^{n} \vec{B}_j \cdot \overrightarrow{\Delta S_j}$$

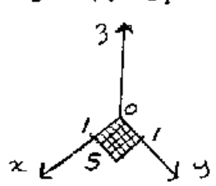
where $\overrightarrow{\Delta S_j} \triangleq \Delta S_j \vec{i}_{nj}$, and $\vec{i}_{nj}$ is the unit vector normal to ΔS_j.

In the limit as n (the number of incremental squares (ΔS) that S is divided up into) $\rightarrow \infty$, we get the exact value of the magnetic flux crossing S:

$$[\phi]_S = \lim_{n\to\infty} \sum_{j=1}^{n} \vec{B}_j \cdot \overrightarrow{\Delta S}_j = \int_S \vec{B} \cdot d\vec{s} = \text{"Surface Integral of } \vec{B} \text{ over the surface } S\text{"}$$

Example: Consider $\vec{B} = 3xy^2 \vec{i}_z$ webers/m^2 and the portion of the xy plane lying between $x=0$, $x=1$, $y=0$, and $y=1$. Find ϕ "cutting" this surface.

$$\vec{B} = 3xy^2 \vec{i}_z \qquad d\vec{s} = dx\,dy\,\vec{i}_z$$



$$\vec{B} \cdot d\vec{s} = (3xy^2 \vec{i}_z) \cdot (dx\,dy\,\vec{i}_z) = 3xy^2\,dx\,dy$$

$$[\phi]_S = \int_S \vec{B} \cdot d\vec{s} = \int_{x=0}^{1} \int_{y=0}^{1} 3xy^2\,dx\,dy$$

(1-21)

$$[\Phi]_s = 3\int_{x=0}^{1} x \left[\frac{y^3}{3}\right]_0^1 dx = \int_0^1 x\, dx = \underline{.5 \text{ webers}}$$

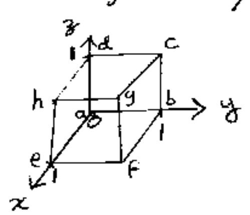
Surface Integration over a Closed Surface:

If the surface integral is evaluated over a closed surface such as the surface of a cube, sphere, or cylinder, then the integral is written as

$$\oint_S \vec{B}\cdot d\vec{s}$$

Example: Determine the amount of time it takes for 1 coulomb of charge to escape from the cubical box below if the current density in the region is given by

$$\vec{J} = (x+2)\,\vec{i}_x + (1-3y)\,\vec{i}_y + 2z\,\vec{i}_z \ \frac{\text{Amp.}}{\text{m}^2}$$



Solution: We wish to find the net current flowing out of the box, in coulombs/second, or amperes. This must be given by

$$\oint_{\substack{\text{SURFACE}\\ \text{OF CUBE}}} \vec{J}\cdot d\vec{s}$$

$$= \int_{\substack{\text{SURFACE}\\ abcd}} \vec{J}\cdot d\vec{s} + \int_{\substack{\text{SURFACE}\\ efgh}} \vec{J}\cdot d\vec{s} + \int_{\substack{\text{SURFACE}\\ aehd}} \vec{J}\cdot d\vec{s} + \int_{\substack{\text{SURFACE}\\ bfgc}} \vec{J}\cdot d\vec{s} + \int_{\substack{\text{SURFACE}\\ aefb}} \vec{J}\cdot d\vec{s} + \int_{\substack{\text{SURFACE}\\ dcgh}} \vec{J}\cdot d\vec{s}$$

(1-22)

Surface abcd (in the plane $x = 0$)

$$\vec{J} = \left[(x+2)\,\vec{i}_x + (1-3y)\,\vec{i}_y + 2z\,\vec{i}_z\right]_{x=0} = 2\,\vec{i}_x + (1-3y)\,\vec{i}_y + 2z\,\vec{i}_z$$

$$d\vec{s} = dy\,dz\,(-\vec{i}_x) = -dy\,dz\,\vec{i}_x$$

$$\int_{abcd} \vec{J}\cdot d\vec{s} = \int_{z=0}^{1}\int_{y=0}^{1} (-2)\,dy\,dz = \underline{-2\ A.}$$

Surface efgh (in the plane $x = 1$)

$$\vec{J} = 3\,\vec{i}_x + (1-3y)\,\vec{i}_y + 2z\,\vec{i}_z\ , \quad d\vec{s} = dy\,dz\,\vec{i}_x$$

$$\int_{efgh} \vec{J}\cdot d\vec{s} = \int_{z=0}^{1}\int_{y=0}^{1} 3\,dy\,dz = \underline{3\ A.}$$

Surface aehd (in the plane $y = 0$)

$$\vec{J} = (x+2)\,\vec{i}_x + \vec{i}_y + 2z\,\vec{i}_z\ , \quad d\vec{s} = -dz\,dx\,\vec{i}_y$$

$$\int_{aehd} \vec{J}\cdot d\vec{s} = \int_{x=0}^{1}\int_{z=0}^{1} (-1)\,dz\,dx = \underline{-1\ A.}$$

Surface bfge (in the plane $y = 1$)

$$\vec{J} = (x+2)\,\vec{i}_x - 2\,\vec{i}_y + 2z\,\vec{i}_z\ , \quad d\vec{s} = dz\,dx\,\vec{i}_y$$

$$\int_{bfge} \vec{J}\cdot d\vec{s} = \int_{x=0}^{1}\int_{z=0}^{1} (-2)\,dz\,dx = \underline{-2\ A.}$$

(1-23)

Surface aefb (in the plane $z = 0$)

$$\vec{J} = (x+2)\,\vec{i}_x + (1-3y)\,\vec{i}_y + 0\,\vec{i}_z\,, \qquad d\vec{s} = -dx\,dy\,\vec{i}_z$$

$$\int_{aefb} \vec{J}\cdot d\vec{s} = 0 \text{ A.}$$

Surface dcgh (in the plane $z = 1$)

$$\vec{J} = (x+2)\,\vec{i}_x + (1-3y)\,\vec{i}_y + z\,\vec{i}_z\,, \qquad d\vec{s} = dx\,dy\,\vec{i}_z$$

$$\int_{dcgh} \vec{J}\cdot d\vec{s} = \int_{y=0}^{1}\int_{x=0}^{1} z\,dx\,dy = 2 \text{ A.}$$

Thus $\displaystyle\oint_S \vec{J}\cdot d\vec{s} = -2 + 3 - 1 - 2 + 0 + 2 = $ 0 Amperes

$\Rightarrow$ **No** net charge will ever escape from this cube!

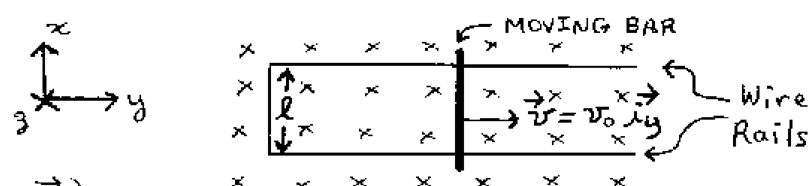
F. FARADAY'S LAW OF INDUCTION (Experimentally Verifiable!)

$$\boxed{\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\overbrace{\int_S \vec{B}\cdot d\vec{s}}^{[\phi]_S}\,,}$$

where S is any surface bounded by the closed path C.

Thus, if a loop of wire is placed in a time-varying magnetic field, an electromotive force, or voltage, is induced in the loop whose magnitude is equal to the time rate of change of the magnetic flux cutting any surface enclosed by the loop. Furthermore, Faraday found that if $d\vec{s}$ is oriented <u>into the paper</u> and the line integral is evaluated in a <u>clockwise direction</u> such that we are consistent with the "right-hand threaded screw rule" (such a screw advances <u>into</u> the paper when the screw is turned clockwise) we find that the voltage comes out negative as $[\emptyset]_s$ increases — Thus the (−) sign (Lenz's Law) is included

<u>Example #1</u>



UNIFORM MAGNETIC FIELD $\vec{B} = B_o \vec{i}_z$

$$[\emptyset]_{\substack{\text{into}\\ \text{paper}}} = \int_S (B_o \vec{i}_z)\cdot(ds\, \vec{i}_z)$$

$$= B_o \int_S ds = B_o\, [\ell\,(y_o + v_o t)]$$

where y_o is the initial position of the bar at $t = 0$

Then from Faraday's Law:

$$[\text{EMF induced}]_{\text{CLOCKWISE}} = -\frac{d}{dt}[\phi]_{\substack{\text{into}\\\text{paper}}}$$

$$= -\frac{d}{dt}\left[B_0\, l\,(y_0 + v_0 t)\right] = -B_0\, l\, v_0$$

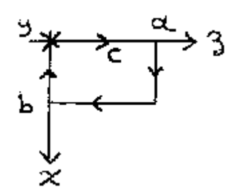
Note that if the bar moves to the right, the EMF produces current in a <u>counter-clockwise sense</u>. This current in turn produces a magnetic field directed out of the paper inside the loop, thus <u>opposing the original magnetic field</u>.

In general, the (−) sign in Faraday's Law is responsible for causing current induced in a closed loop to flow in the direction that sets up a magnetic field that <u>opposes</u> any change in the original magnetic field.

<u>Example # 2</u> Consider a time-varying magnetic field

$$\vec{B} = B_0 \cos\omega t\, \vec{i_y}$$

and a fixed rectangular loop shown below:



$$[\phi]_{\substack{\text{into}\\\text{paper}}} = \int_S \vec{B}\cdot ds\, \vec{i_y}$$

$$= \int_{y=0}^{b}\int_{x=0}^{a} \cos\omega t\, \vec{i_y}\cdot dx\,dz\, \vec{i_y}$$

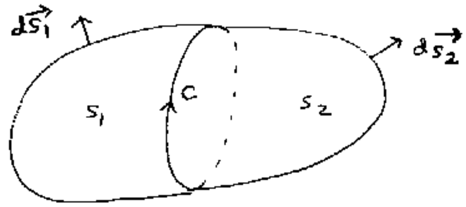
$$= B_0 \cos\omega t \int_{y=0}^{b}\int_{x=0}^{a} dx\, dz = ab\, B_0 \cos\omega t$$

$$[\text{EMF induced}] = -\frac{d}{dt}[\phi]_{\substack{\text{into}\\\text{paper}}} = ab B_0 \omega \sin\omega t$$

(1-25)

G. Gauss' Law for the Magnetic Field

Consider a closed path C and two surfaces S_1 and S_2 both of which are enclosed (bounded) by C.



Then Faraday's Law $\Rightarrow$

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_{S_1} \vec{B}\cdot d\vec{S_1}$$

and

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_{S_2} \vec{B}\cdot\left(-d\vec{S_2}\right) = +\frac{d}{dt}\int_{S_2} \vec{B}\cdot d\vec{S_2}$$

$$\Rightarrow \quad \frac{d}{dt}\int_{S_2} \vec{B}\cdot d\vec{S_2} = -\frac{d}{dt}\int_{S_1} \vec{B}\cdot d\vec{S_1}$$

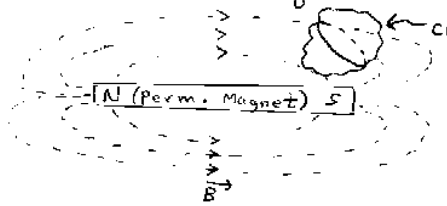
$$\frac{d}{dt}\left[\int_{S_1} \vec{B}\cdot d\vec{S_1} + \int_{S_2} \vec{B}\cdot d\vec{S_2}\right] = \frac{d}{dt}\oint_{S_1+S_2} \vec{B}\cdot d\vec{s} = 0$$

$$\oint_{S_1+S_2} \vec{B}\cdot d\vec{s} = \text{Constant w.r.t. time}$$

But since there is little evidence of the above, we state that

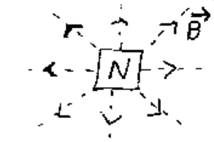
$$\boxed{\oint_S \vec{B}\cdot d\vec{s} = 0 \quad \text{(Gauss' Law for Magnetic Field)}}$$

Gauss' Law for the magnetic field states that the net magnetic flux entering a region space bounded by a closed surface must equal the flux leaving this region.



We see that the field's set up by magnetic objects ("dipoles") (like permanent or electro- magnets) having both North & South poles will always obey this rule – no matter where the closed surface is placed. If we could find a magnetic "monopole", say, an object that originates flux but does not "receive it" again, we could place this monopole inside a closed "Gaussian Surface" and have

Magnetic Monopole



$$\oint_S \vec{B}\cdot d\vec{s} = \text{Constant}$$

as predicted by Faraday's Law!

H. Ampere's Circuital Law

$$\oint_C \frac{\vec{B}}{\mu_0}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_S \epsilon_0 \vec{E}\cdot d\vec{s}$$

where S is any surface bounded by C.

Note that the same surface must be employed to evaluate the integrals on the right hand side of the equation, and that the right-hand screw rule must be applied to fix the direction C is traversed relative to the choice of orientation of $d\vec{s}$ (is $d\vec{s}$ oriented "out of" or "into" paper?).

To simplify the statement of Ampere's Circuital Law, let us define two new fields:

Magnetic Field Intensity ($\vec{H}$)

$$\boxed{\vec{H} \triangleq \frac{\vec{B}}{\mu_0}}$$

(This is the relationship relating $\vec{H}$ and $\vec{B}$ in "free space.")
Note $\vec{H}$ is in the direction of $\vec{B}$.

Recall the Biot-Savart Law

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \vec{R}}{R^3} \Rightarrow \vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{4\pi} \frac{\vec{I}\, d\vec{\ell}}{R^2} \times \vec{i}_R \quad \frac{\text{Amp} \cdot \text{meter}}{(\text{meter})^2}$$

Thus the units of $\vec{H}$ are amperes/meter.

$\oint_C \vec{H} \cdot d\vec{\ell}$ is called the magnetomotive force (mmf) by analogy with $\oint_C \vec{E} \cdot d\vec{\ell}$, the electromotive force (emf).

(1-29)

However, mmf does not mean that it is the work done moving a charge around closed path C in the presence of a magnetic field, since the force exerted by a magnetic field on a moving charge is directed normal to the direction of motion of the charge!

Electrical Displacement Flux Density ($\vec{D}$)

$$\boxed{\vec{D} = \epsilon_0 \vec{E}}$$

Again, this defines D in free space.
Note $\vec{D}$ is in the direction of $\vec{E}$.

Recall Coulomb's Law

$$\vec{F} = q\vec{E} = \frac{qQ\, \vec{i}_R}{4\pi\epsilon_0 R^2} \Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} \vec{i}_R \left(\frac{\text{Newt.}}{\text{Coul}}\right)$$

$$\therefore \quad \vec{D} = \epsilon_0 \vec{E} = \frac{Q}{4\pi R^2} \vec{i}_R \quad \frac{\text{coul}}{(\text{meter})^2}$$

Thus the units of the $\vec{D}$ field are $\underline{\text{coul}/(\text{meter})^2}$.

Note $\int_S \vec{D} \cdot d\vec{s}$ = displacement flux crossing S, whose units are $[\text{coul}/(\text{meter})^2] \cdot (\text{meter})^2$ = coulombs.

Hence $\frac{d}{dt} \int_S \vec{D} \cdot d\vec{s}$ = time rate of change of $\vec{D}$, whose units must come out to be coulombs/second = <u>amperes</u>

Thus $\frac{d}{dt} \int_S \vec{D} \cdot d\vec{s}$ is sometimes called "<u>displacement current</u>"

Note that displacement current can exist in a charge-free region (free space) thus it does <u>NOT</u> represent a physical

(1-30)

flow of charges! Physical charge flow is represented by $\int_S \vec{J} \cdot d\vec{s}$ = current crossing S due to actual motion of charges.

Thus, <u>Ampere's Circuital Law</u> becomes

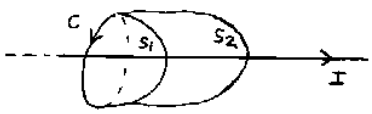
$$\boxed{\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{s} + \frac{d}{dt} \int_S \vec{D} \cdot d\vec{s}}$$

$\Rightarrow$ "The magnetomotive force around a closed path C is equal to the "<u>total current</u>", that is, the sum of the current due to the actual flow of charges and the "displacement current" bounded by C, in accordance with the right-hand screw rule."

Loosely speaking, the concept of <u>displacement current</u>, which is <u>not</u> actual charge flow, simply is a way of indicating that a changing electric field sets up a magnetic field in much the same way that a flow of charges does!

3 Cases Illustrating Ampere's Circuital Law

(a) <u>Infinitely Long, Current-Carrying Wire</u>:



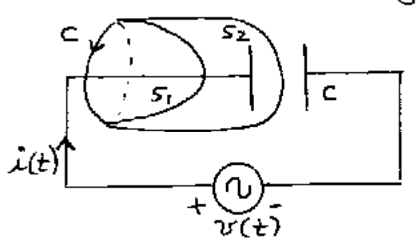
(No displacement flux present)

$$\int_{S_1} \vec{J}\cdot d\vec{s} = \int_{S_2} \vec{J}\cdot d\vec{s} = I$$

$$\therefore \oint_C \vec{H}\cdot d\vec{\ell} = I$$

(1-31)

(b) <u>Circuit Containing a Capacitor</u>

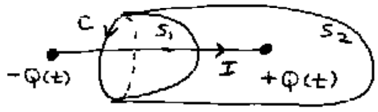


$$\int_{S_1} \vec{J}\cdot d\vec{s} = i \quad \underline{\text{but}} \quad \int_{S_2} \vec{J}\cdot d\vec{s} = 0$$

$$\int_{S_1} \vec{D}\cdot d\vec{s} = 0 \quad \underline{\text{but}} \quad \int_{S_2} \vec{D}\cdot d\vec{s} \neq 0$$

In fact, $\frac{d}{dt}\int_{S_2} \vec{D}\cdot d\vec{s}$ must $= i$, so that $\oint_C \vec{H}\cdot d\vec{\ell} = i$ regardless of whether S_1 or S_2 is chosen as the enclosed surface.

(c) Finitely Long Wire



$\int_{S_1} \vec{J}\cdot d\vec{s} = I$ and $\frac{d}{dt}\int_{S_1}\vec{D}\cdot d\vec{s} \neq 0$

$\int_{S_2} \vec{J}\cdot d\vec{s} = 0$ and $\frac{d}{dt}\int_{S_2}\vec{D}\cdot d\vec{s} \neq 0$

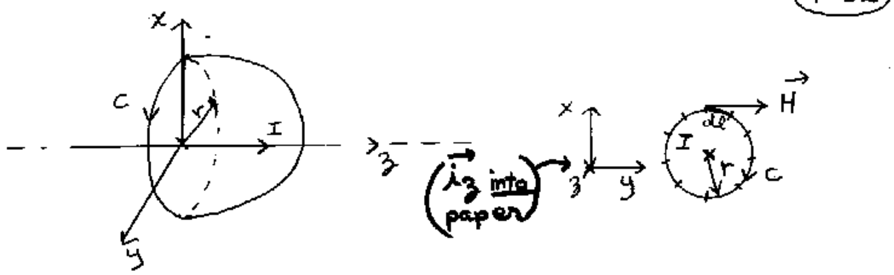
Note $\int_{S_1} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{S_1}\vec{D}\cdot d\vec{s} = \frac{d}{dt}\int_{S_2}\vec{D}\cdot d\vec{s}$

so that $\int_C \vec{H}\cdot d\vec{\ell}$ is unique regardless of whether S_1 or S_2 is chosen as the enclosed surface.

Example #1 Consider an infinitely long, thin, straight wire along the z axis and carrying current in the z direction.

Find the value of $\vec{H}$ in the region about this wire.

(1-32)



If we choose a circular contour C situated in the xy plane, we have

$$\oint \vec{H}\cdot d\vec{\ell} = I$$

Furthermore, we know from elementary consideration of the Biot-Savart law and symmetry that $\vec{H}$ must be directed tangentially to the circular path and must have uniform magnitude at points along this path.

Thus, if the path is subdivided into infinitesimally long segments, then for each segment

$$\vec{H} \cdot d\vec{l} = H\,dl$$

and
$$\oint_C \vec{H} \cdot d\vec{l} = \oint_C H\,dl = H \oint_C dl = 2\pi r \cdot H$$

Ampere's Circuital Law $\Rightarrow$

$$\oint_C \vec{H} \cdot d\vec{l} = 2\pi r H = I \quad \Rightarrow \quad \boxed{H = \frac{I}{2\pi r}}$$

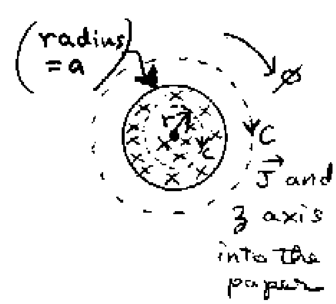
Thus the magnetic field due to an infinitely long wire is directed circular to the wire in the right-hand sense, and has a magnitude equal to $I/2\pi r$, where r is the distance from the point to the wire.

$$\mathbf{H} = H_\Phi \mathbf{i}_\Phi = [\,I/(2\,\pi\, r)]\,\mathbf{i}_\Phi$$

1-33

Example # 2 We now extend the previous example to the case of an infinitely long, straight, cylindrical conductor of radius a, carrying uniformly distributed current. Let the conductor be concentric with the z axis such that

$$\vec{J} = \begin{cases} J_0\,\vec{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$



From symmetry considerations,

$$\vec{H} = H_\phi(r)\,\vec{i}_\phi$$

That is, $\vec{H}$ must be directed everywhere circular to the axis of the wire and its magnitude must be uniform on cylindrical surfaces coaxial with the cylindrical wire. Then if we choose a <u>circular path</u> of radius r in a plane normal to the wire and centered on the axis of the wire, we have

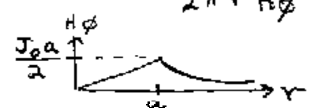
$$\oint_C \vec{H}\cdot d\vec{\ell} = 2\pi r\, H_\phi$$

<u>and</u>

$$\int_S \vec{J}\cdot d\vec{s} = \begin{cases} J_0 \pi r^2 & \text{if } C \text{ inside wire} \Rightarrow r \le a \\ J_0 \pi a^2 & \text{if } C \text{ outside wire} \Rightarrow r > a \end{cases}$$

$\therefore$ by Ampere's Circuital Law

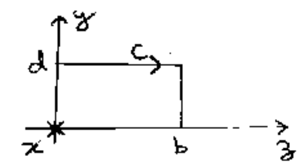
$$2\pi r\, H_\phi = \begin{cases} J_0 \pi r^2 & \text{for } r \le a \\ J_0 \pi a^2 & \text{for } r > a \end{cases} \Rightarrow \boxed{H_\phi = \begin{cases} \frac{J_0 r}{2}, & r \le a \\ \frac{J_0 a^2}{2r}; & r > a \end{cases}}$$



(1-34)

<u>Example #3</u> Consider a rectangular loop in the yz plane immersed in a time-varying electric field,

$$\vec{E} = E_0\, z \sin\omega t\, \vec{i_x}.$$



Then, by Ampere's Circuital Law,

$$\oint \vec{H}\cdot d\vec{\ell} = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s}$$

$$\int_S \vec{D}\cdot d\vec{s} = \int_{z=0}^{b}\int_{y=0}^{d} \epsilon_0 E_0 z \sin\omega t\, \vec{i_x} \cdot dy\, dz\, \vec{i_x}$$

$$= \epsilon_0 E_0 \sin\omega t \int_{z=0}^{b}\int_{y=0}^{d} z\, dy\, dz = \epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t$$

$$\oint_C \vec{H}\cdot d\vec{\ell} = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s} = \frac{d}{dt}\left[\epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t\right]$$

$$\int_S \vec{D}\cdot d\vec{s} = \int_{z=0}^{b}\int_{y=0}^{d} \epsilon_0 E_0 z \sin\omega t\, \vec{i_x} \cdot dy\, dz\, \vec{i_x}$$

$$= \epsilon_0 E_0 \sin\omega t \int_{z=0}^{b}\int_{y=0}^{d} z\, dy\, dz = \epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t$$

$$\oint_C \vec{H}\cdot d\vec{l} = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s} = \frac{d}{dt}\left[\epsilon_0 \frac{b^2 d}{2} E_0 \sin\omega t\right]$$

$$\oint_C \vec{H}\cdot d\vec{l} = \epsilon_0 \frac{b^2 d}{2} E_0 w \cos\omega t$$

Can we determine $\vec{H}$ now? – No! because $\vec{H}$ is not uniform over the closed path (rectangular loop), C.

(1-35)

I. Gauss' Law for the Electric Field

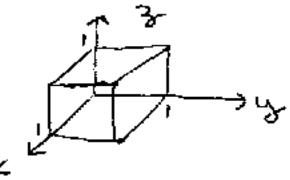
$$\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv$$

$\oint_S \vec{D}\cdot d\vec{s} \triangleq$ Displacement flux emanating from the closed surface S i.e., the flow of $\vec{D}$ or $\epsilon_0\vec{E}$

$$\rho \triangleq \text{Volume charge density} = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v},\ \text{C/m}^3$$

Thus $\int_V \rho\, dV$ = Volume Charge Contained in V

Example of volume integration



Assume $\rho = x+y+z \ \ C/m^3$

To find the charge within the cube

$$\int_V \rho\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left[xz+yz+\frac{z^2}{2}\right]_{z=0}^{1} dx\,dy$$

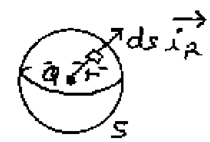
$$= \int_{x=0}^{1}\left[xy+\frac{y^2}{2}+\frac{y}{2}\right]_{y=0}^{1} dx = \int_{x=0}^{1}(x+1)\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_0^1 = 1.5 \text{ Coulombs}$$

Thus the displacement flux, $\oint_s \vec{D}\cdot d\vec{s}$, emanating from the cubical surface is 1.5 coul.

Gauss' Law is often used to compute static fields.

Example #1 Finding field due to point charge via Gauss' Law:



From symmetry considerations,

$$\vec{E} = E_r(r)\,\vec{i}_R$$

$$\vec{D} = \epsilon_0\vec{E} = D_r(r)\,\vec{i}_r$$

Then, if we choose a (Gaussian) surface that is a sphere of radius r, over which D is uniform, and then apply Gauss' Law over this surface,

$$\oint_S \vec{D}\cdot d\vec{s} = \oint_S (D_r\,\vec{i}_R)\cdot(ds\,\vec{i}_R) = D_r \oint_S ds$$

$$= (D_r)\,(\text{area of sphere}) = D_r\,(4\pi r^2)$$

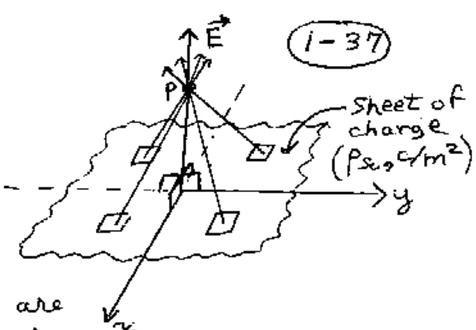
$$\int_V \rho \, dV = \text{charge enclosed by } S = \text{point charge, } Q.$$

Thus $\quad 4\pi r^2 D_r = Q$

$$D_r = \frac{Q}{4\pi r^2}, \quad E_r = \frac{D_r}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{i}_R$$

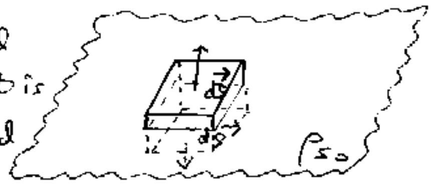
<u>Example #2</u> Finding field due to an infinite sheet of charge in the xy plane, having uniform surface charge density ρ_{so} (c/m^2). We desire to find $\vec{E}$ everywhere.



We first note that $\vec{E}$ must be directed solely in the z direction, since for any infinitesimal patch of charge on the sheet $\rho_s \, ds$, there are 3 other symmetrically located charge patches which produces a net charge contribution at the observation point, P, that is directed solely along $\vec{i}_z$. Likewise, since this argument can be made for any position, P, above the x, y plane, we can argue that $\vec{E}$ will be independent of x, y. Of course, the field below the sheet must likewise be directed in the $-z$ direction. Thus, we can say that

$$\vec{D} = \begin{cases} D_z(z)\,\vec{i_z} & \text{for } z>0 \\ -D_z(z)\,\vec{i_z} & \text{for } z<0 \end{cases}$$

Consider a differential rectangular box that is symmetrically situated about the sheet of charge. Applying Gauss' law:



$$\oint_S \vec{D}\cdot d\vec{s} = \underset{\text{top of box}}{\int \vec{D}\cdot d\vec{s}} + \underset{\text{bottom of box}}{\int \vec{D}\cdot d\vec{s}} + \underset{\text{4 sides of box}}{\int \vec{D}\cdot d\vec{s}}$$

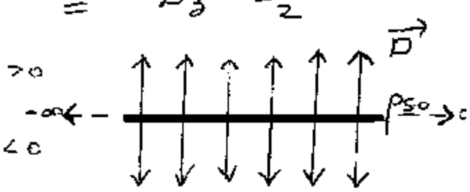
$$= D_z\, ds + D_z\, ds + 0 = 2D_z\, ds$$

top & bottom have surface area ds

$$\int_V \rho\, dv = \text{surface charge enclosed by the box} = \rho_{so}\, ds$$ (1-38)

Thus, $2D_z\, ds = \rho_{so}\, ds$ or $D_z = \frac{\rho_{so}}{2}$

$$\vec{D} = \begin{cases} \frac{\rho_{so}}{2}\,\vec{i_z} & \text{for } z>0 \\ \frac{-\rho_{so}}{2}\,\vec{i_z} & \text{for } z<0 \end{cases}$$

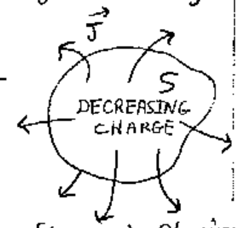


SOMEWHAT SURPRISINGLY, $\vec{E}$ & $\vec{D}$ are not dependent on the height above the ∞ extended sheet of charge!

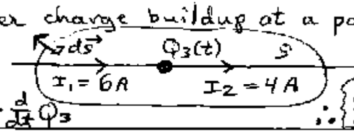
Derivation of Gauss' Law from Ampere's Circuital Law

Law of Conservation of charge: The current due to flow of charges emanating from a closed surface, S, is equal to the time rate of decrease of the charge in the volume V enclosed by S,

Conservation of Charge: $$\oint_S \vec{J}\cdot d\vec{s} = -\frac{d}{dt}\int_V \rho\, dv$$

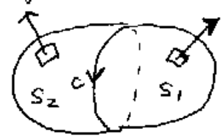


Or, consider charge buildup at a point in a wire:



$\Rightarrow -6A + 4A = -\frac{d}{dt}Q_3$ $\therefore \frac{dQ_3}{dt} = 2A$

Now, consider two surfaces, S_1 and S_2 bounded by the same closed path C.



Ampere's Circuital Law $\Rightarrow$

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{S_1} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{S_1} \vec{D}\cdot d\vec{s}$$

$$\oint_C \vec{H}\cdot d\vec{l} = -\int_{S_2} \vec{J}\cdot d\vec{s} - \frac{d}{dt}\int_{S_2} \vec{D}\cdot d\vec{s}$$

Thus, we may equate the right hand sides of these two expressions for $\oint_C \vec{H}\cdot d\vec{l}$: (1-39)

$$\int_{S_1} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{S_1} \vec{D}\cdot d\vec{s} = -\int_{S_2} \vec{J}\cdot d\vec{s} - \frac{d}{dt}\int_{S_2} \vec{D}\cdot d\vec{s}$$

$$\overset{\text{or}}{=} \oint_{S_1+S_2} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\oint_{S_1+S_2} \vec{D}\cdot d\vec{s} = 0$$

$$\underbrace{-\frac{d}{dt}\int_V \rho\, dV}_{\text{by Law of Charge Conservation}} + \frac{d}{dt}\oint_S \vec{D}\cdot d\vec{s} = 0$$

$$\frac{d}{dt}\left[\oint_S \vec{D}\cdot d\vec{s} - \int_V \rho\, dv\right] = 0$$

$$\therefore \oint_S \vec{D}\cdot d\vec{s} - \int_V \rho\, dv = \text{const. w.r.t time} = 0$$

$$\therefore \oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv$$, which is Gauss' Law!

In summary (From Wikipedia!)

Name	**Differential form**	**Integral form**
Gauss's law:	$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{A} = \int_V \rho dV$
Gauss' law for magnetism (absence of magnetic monopoles):	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}$
Ampère's law (with Maxwell's extension):	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{A} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{A}$