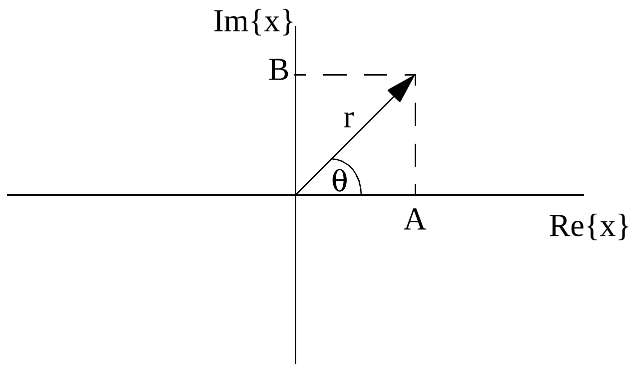
Complex Number Refresher

EE's use $j = \sqrt{-1}$ instead of i, so that we don't confuse current, which we often label as i, with imaginary numbers

A complex number $\underline{x} = A + jB$, note the underlining to denote a complex number.

- A bold font is also often used to denote complex numbers.
- it has a real part Re{x}=A
- it has an imaginary part Im{x}=B

Complex numbers can be drawn as vectors on the COMPLEX PLANE:



As a vector, we can speak of a complex number in terms of it's magnitude, r, and angle, θ.

- $\underline{x} = A + jB = r\angle\theta$
- $r = |\underline{x}|$

To convert between polar and cartesian coordinates:

- If you have cartesian form $\underline{x} = A + jB$
  - $r = |\underline{x}| = \sqrt{A^2 + B^2}$
  - $\theta = \text{atan}\left(\frac{B}{A}\right)$
- if you have polar form $\underline{x} = r\angle\theta$
  - $A = r\cos(\theta)$
  - $B = r\sin(\theta)$

You MUST use "four-quadrant" arctangent, since the standard "ATAN" button on your calculator does only a "two-quadrant" arctangent, meaning that the returned angle is always between -90 and +90 degrees (i.e., quadrants 1 and 4). The reason is that the signs of A and B introduce an ambiguity; 2+j2 and -2-j2 both have a positive ratio B/A, so simply doing ATAN on -2/-2 will erroneously yield an angle of +45 degrees.

The best solution is to learn how to use polar-to-rectangular conversion functions on your calculator, or even better, the complex numbers facilities.

There is another form that we use called Exponential Form

- $re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$ see Euler's Formula on page 158 of textbook
- This is just a complex number with polar form, $r\angle\theta$, and cartesian form, $r\cos(\theta) + jr\sin(\theta)$

So now we have $\underline{x} = A + jB = r\angle\theta = re^{j\theta}$. But WHY do we care about these different forms?

ANSWER = Complex Arithmetic

When doing addition and subtraction of complex numbers use **Cartesian Form**:

- $\underline{x}_1 = A + jB$ and $\underline{x}_2 = C + jD$
  - $\underline{x}_1 + \underline{x}_2 = (A + C) + j(B + D)$
  - $\underline{x}_1 - \underline{x}_2 = (A - C) + j(B - D)$
- Example $\underline{x}_1 = -3 + j6$ and $\underline{x}_2 = 2 - j3$
  - $\underline{x}_1 + \underline{x}_2 = -1 + j3$
  - $\underline{x}_1 - \underline{x}_2 = -5 + j9$

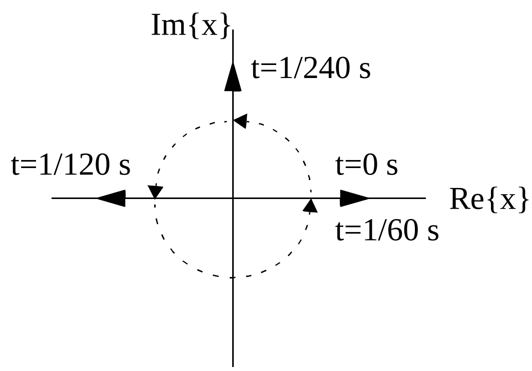
When doing multiplication and division of complex numbers use **Polar Form**

- $\underline{x}_1 = r_1\angle\theta_1$ and $\underline{x}_2 = r_2\angle\theta_2$
  - $\underline{x}_1\underline{x}_2 = r_1 r_2\angle(\theta_1 + \theta_2) = r_1 e^{j\theta_1}(r_2 e^{j\theta_2}) = r_1 r_2 e^{j\theta_1} e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
  - $\underline{x}_1 / \underline{x}_2 = (r_1/r_2)\angle(\theta_1 + \theta_2) = \dfrac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \dfrac{r_1}{r_2}(e^{j\theta_1} e^{-j\theta_2}) = \dfrac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
- Example $\underline{x}_1 = 3\angle 10°$ and $\underline{x}_2 = -2\angle -5°$
  - $\underline{x}_1\underline{x}_2 = -6\angle 5°$
  - $\underline{x}_1 / \underline{x}_2 = -1.5\angle 15°$

We represent AC voltages and currents with sinusoids: $A\cos(\omega t)$

- $A\cos(\omega t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$ see Euler's Formula on page 158 of Text
- $Ae^{j\omega t}$ is a complex number in exponential form, but what does this mean?
- A is simply the magnitude of the vector, so what is $e^{j\omega t}$?
- If we convert this to polar form we have $e^{j\omega t} = 1\angle\omega t$, which is a unit vector with an angle of ωt. Let's assume that $f = 60\text{Hz}$ so that $\omega = 2\pi f = 120\pi$ and see what $e^{j\omega t} = 1\angle\omega t$ looks like in the complex plane...

t	ωt
0	0
1/240 s	π/2
1/120 s	π
1/60 s	2π

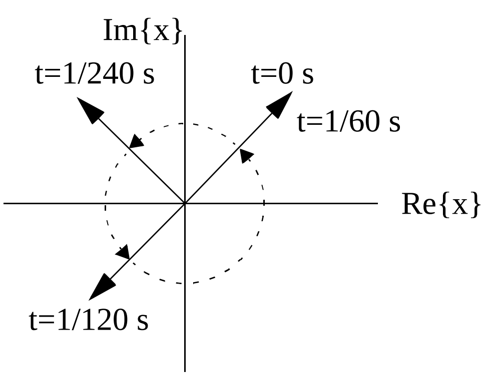


- So what we have is a vector that rotates around the complex plane at 60 times per second.

Sometimes the AC voltages and currents have a phase shift: $A\cos(\omega t + \theta)$

- $A\cos(\omega t + \theta) = \frac{A}{2}e^{j(\omega t + \theta)} + \frac{A}{2}e^{-j(\omega t + \theta)}$
- How does this relate to the example above?
- Let's assume the same frequency as before, but now we have a phase shift of $\theta = \pi/4$.

t	ωt+π/4
0	π/4
1/240 s	3π/4
1/120 s	5π/4
1/60 s	9π/4



- Produces the same rotating vectors as before except that at time t=0 s the vectors start at +45 degrees from the Real Axis.