Trigonometry Review for Signals and Systems

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Introduction

There is a small set of trigonometric identities that are used repeatedly in signal and systems analysis. This review lists these identities and shows how they can be quickly derived when needed.

It is tempting to imagine that formulas such as trig identities can always be looked up somewhere when you need them. Often, however, you will need the fomulas at times when access to textbooks or handbooks is impossible. You will be at lunch with a colleague doing calculations on a napkin. You will be on an airplane. You will be in a meeting with clients. You will be in a hotel room preparing a presentation for the next morning. You will be setting up an installation at a field site. If the formulas are not in your head, they will be altogether unavailable. On the other hand, it is difficult to memorize an extensive table of formulas that are all nearly alike, and then keep straight where all of the sines and cosines and plusses and minusses go. The solution is to memorize one or two basic formulas, and learn to derive the others quickly when you need them.

The Basics

The basic trigonometric formula, and one you should memorize, is actually a pair of identities called Euler's identities. These are:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{1.1}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2} \tag{1.2}$$

Euler's identities are usually proved by expanding each side in a power series, and showing that these series are identical. We will omit this proof, and accept the identities as our starting point.

If equation (1.2) is multiplied by j and added to equation (1.1) we obtain

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{1.3}$$

If we subtract rather than add we obtain

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \tag{1.4}$$

Equations (1.3) and (1.4) are also sometimes called Euler's identities.

The Derived Identities

Product of Cosines

The following identities can be derived by substituting for the cosines and sines using Euler's identities. The first case involves the product of two cosines.

$$\cos(\theta)\cos(\phi) = \frac{e^{j\theta} + e^{-j\theta}}{2}\frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$= \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi} + e^{j\theta}e^{-j\phi} + e^{-j\theta}e^{j\phi}}{4}$$

$$= \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi}}{4} + \frac{e^{j\theta}e^{-j\phi} + e^{-j\theta}e^{j\phi}}{4}$$

$$= \frac{e^{j(\theta+\phi)} + e^{-j(\theta+\phi)}}{4} + \frac{e^{j(\theta-\phi)} + e^{-j(\theta-\phi)}}{4}$$

Using Euler's identity again gives the result

$$\cos(\theta)\cos(\phi) = \tfrac{1}{2}\cos(\theta+\phi) + \tfrac{1}{2}\cos(\theta-\phi) \tag{1.5}$$

Product of Sines

Proceeding as in the above derivation,

$$\sin(\theta)\sin(\phi) = \frac{e^{j\theta} - e^{-j\theta}}{j2}\frac{e^{j\phi} - e^{-j\phi}}{j2}$$

$$= \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi} - e^{j\theta}e^{-j\phi} - e^{-j\theta}e^{j\phi}}{-4}$$

$$= -\frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi}}{4} - \frac{-e^{j\theta}e^{-j\phi} - e^{-j\theta}e^{j\phi}}{4}$$

$$= -\frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi}}{4} + \frac{e^{j\theta}e^{-j\phi} + e^{-j\theta}e^{j\phi}}{4}$$

$$= -\frac{e^{j(\theta+\phi)} + e^{-j(\theta+\phi)}}{4} + \frac{e^{j(\theta-\phi)} + e^{-j(\theta-\phi)}}{4}$$

giving the result

$$\sin(\theta)\sin(\phi) = -\tfrac{1}{2}\cos(\theta+\phi) + \tfrac{1}{2}\cos(\theta-\phi) \tag{1.6}$$

Product of Sine and Cosine

This is another variation on the same theme.

$$\sin(\theta)\cos(\phi) = \frac{e^{j\theta} - e^{-j\theta}}{j2}\frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$= \frac{e^{j\theta}e^{j\phi} - e^{-j\theta}e^{-j\phi} + e^{j\theta}e^{-j\phi} - e^{-j\theta}e^{j\phi}}{j4}$$

$$= \frac{e^{j\theta}e^{j\phi} - e^{-j\theta}e^{-j\phi}}{j4} + \frac{e^{j\theta}e^{-j\phi} - e^{-j\theta}e^{j\phi}}{j4}$$

$$= \frac{e^{j(\theta+\phi)} - e^{-j(\theta+\phi)}}{j4} + \frac{e^{j(\theta-\phi)} - e^{-j(\theta-\phi)}}{j4}$$

Using (1.2) gives

$$\sin(\theta)\cos(\phi) = \tfrac{1}{2}\sin(\theta+\phi) + \tfrac{1}{2}\sin(\theta-\phi) \tag{1.7}$$

Equal Angles

The above formulas reduce to specialized results when $\theta = \phi$. From (1.5) we obtain

$$\cos^2(\theta) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta) \qquad (1.8)$$

From (1.6) we get

$$\sin^2(\theta) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) \qquad (1.9)$$

Finally, from (1.7) we have

$$\sin(\theta)\cos(\theta) = \tfrac{1}{2}\sin(2\theta) \qquad (1.10)$$

Cosine of Sum and Difference

This identity can be easily derived by subtracting (1.6) from (1.5), but it will be derived here from Euler's identities for consistency.

$$\begin{aligned}
\cos(\theta+\phi) &= \frac{e^{j(\theta+\phi)} + e^{-j(\theta+\phi)}}{2} \\
&= \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi}}{2} \\
&= \frac{2e^{j\theta}e^{j\phi} + 2e^{-j\theta}e^{-j\phi}}{4} \\
&= \frac{2e^{j\theta}e^{j\phi} + 2e^{-j\theta}e^{-j\phi} + e^{j\theta}e^{-j\phi} - e^{j\theta}e^{-j\phi} + e^{-j\theta}e^{j\phi} - e^{-j\theta}e^{j\phi}}{4} \\
&= \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi} + e^{j\theta}e^{-j\phi} + e^{-j\theta}e^{j\phi}}{4} + \frac{e^{j\theta}e^{j\phi} + e^{-j\theta}e^{-j\phi} - e^{j\theta}e^{-j\phi} - e^{-j\theta}e^{j\phi}}{4} \\
&= \frac{\left(e^{j\theta} + e^{-j\theta}\right)\left(e^{j\phi} + e^{-j\phi}\right)}{4} - \frac{\left(e^{j\theta} - e^{-j\theta}\right)\left(e^{j\phi} - e^{-j\phi}\right)}{-4} \\
&= \frac{\left(e^{j\theta} + e^{-j\theta}\right)\left(e^{j\phi} + e^{-j\phi}\right)}{2\cdot 2} - \frac{\left(e^{j\theta} - e^{-j\theta}\right)\left(e^{j\phi} - e^{-j\phi}\right)}{j2\cdot j2}
\end{aligned}$$

The result is

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \qquad (1.11)$$

The difference identity can be produced by adding (1.6) and (1.5) or by Euler's identity as above. The result is

$$\cos(\theta-\phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) \qquad (1.12)$$

Sine of Sum and Difference

These identities can be easily derived from (1.7), or by Euler's identity as was done above for the cosine of sum identity. The details are left as an exercise for the reader. The results are,

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \qquad (1.13)$$

and

$$\sin(\theta-\phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi) \qquad (1.14)$$

This completes the set of trig identities commonly encountered in signal processing problems.