Solving Recurrence Relations with the Master Theorem

Purpose of Analysis

- When analyzing algorithms, we only care about the asymptotic behavior.
- Rather than solve exactly the recurrence relation associated with the cost of an algorithm, it is enough to give an asymptotic characterization.
- The master theorem is the main tool.

The Master Theorem

- Let $T(n)$ be a monotonically increasing function that satisfies:

  $T(n) = a\,T(n/b) + f(n)$

  $T(1) = c$

  where $a \geq 1$, $b \geq 2$. If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then:

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

The Master Theorem Applied to Known Recurrence Relations

- Linear Search:

  $T(n) = T(n\text{-}1) + \Theta(1)$
- We are not dividing, so we cannot use the Master theorem.

The Master Theorem Applied to Known Recurrence Relations

- Binary Search:

  $T(n) = T(n/2) + \Theta(1)$
  - a = 1
  - b = 2
  - d = 0
- Second form: $T(n) = \Theta(n^0 \log n) = \Theta(\log n)$

The Master Theorem Applied to Known Recurrence Relations

- Worst case of Mergesort :

  $T(n) = 2T(n/2) + \Theta(n)$
  - a = 2
  - b = 2
  - d = 1
- Second form: $T(n) = \Theta(n^1 \log n) = \Theta(n \log n)$

The Master Theorem Applied to Known Recurrence Relations

- Best case of Mergesort :
  $T(n) = 2T(n/2) + \Theta(n)$
  - a = 2
  - b = 2
  - d = 1
- Second form: $T(n) = \Theta(n^1 \log n) = \Theta(n \log n)$

The Master Theorem Applied to Known Recurrence Relations

- Worst case of Quicksort:
  $T(n) = T(n\text{-}1) + \Theta(n)$
- We are not dividing, so we cannot use the Master theorem.

The Master Theorem Applied to Known Recurrence Relations

- Average case of Quicksort :

  T(N) = 2(T(0) + T(1) + T(2) + … + T(N - 1))/N + N
- We are not dividing, so we cannot use the Master theorem.
- Notice that even for the recurrence relation adapted for telescoping that is shown below, we cannot use the Master theorem.

  T(N)/(N+1) = T(N-1)/N + 2/(N+1)

The Master Theorem Applied to Known Recurrence Relations

- Consider:

  $T(n) = 3T(n/2) + n^2$
  - a =
  - b =
  - d =

The Master Theorem Applied to Known Recurrence Relations

- Consider:

  $T(n) = 16T(n/4) + n$

  - a =
  - b =
  - d =

Limitations

- You *cannot* use the Master Theorem if
  - $T(n)$ is not monotone, ex: $T(n) = \sin n$
  - $f(n)$ is not a polynomial, ex: $T(n) = 2T(n/2) + 2^n$
  - b cannot be expressed as a constant, ex: $T(n) = T(\sqrt{n})$
- Note that the Master Theorem does not solve a recurrence relation.