Kinematics

As described earlier in the class kinematics is "the geometry of the motion". Thus far we have only really talked about particles. We are now ready to discuss the kinematics of rigid bodies. We will limit our discussion to planar motion. In the following discussion a "dot" over a symbol implied differentiation of the quantity with respect to time.

Symbol	Meaning
$\vec{r}$	Position vector
r	magnitude of position vector, i.e. distance from the origin
θ	angle position vector makes with the real axis, i.e. argument of position
$\vec{r}_{P/O}$	Position of point P with respect to point O
$\vec{v}$	Velocity vector
$\vec{v}_{P/O}$	Velocity of point P with respect to point O
$\vec{\omega}$	Angular velocity vector
$\vec{\alpha}$	Angular acceleration vector

Basic Kinematics

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \tag{1}$$

and

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} \tag{2}$$

The quantities we use to describe angular motion are angular position, $\vec{\theta}$, angular velocity, $\vec{\omega}$, and angular acceleration, $\vec{\alpha}$. Analogous equation to Eq. (1-2) can be written for these quantities.

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} = \dot{\vec{\theta}} \tag{3}$$

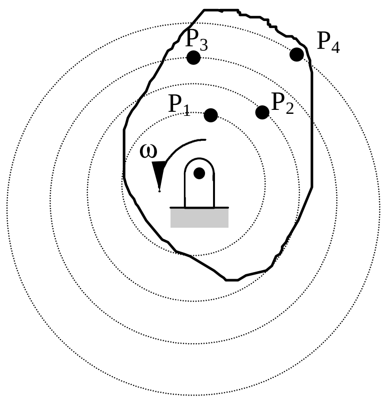
and

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\vec{\omega}} \tag{4}$$

The direction of these vectors are perpendicular to the plane of motion where the direction is given by the right hand rule (curl your fingers in the direction of the motion and your thumb gives you the direction of the vector).

Fixed Axis Rotation

The description is simple: the rigid body has a hinge, joint, or pivot which is connected to a non-moving foundation. The rigid body rotates about a stationary axis passing through this fixed point. There is one point on the rigid body that has zero velocity, and it is of course this fixed point. All other points belonging to the rigid body move in circular arcs about the fixed point. The path of four points on a rigid body undergoing fixed axis rotation is shown below.

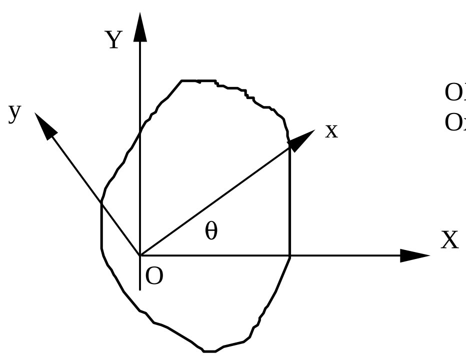


The plane of the motion is perpendicular to the fixed axis.

Mathematics

There are a number of ways to derived the equation for the velocity of any point on a rigid body that is undergoing fixed axis rotation.

<u>Vector differentiation approach.</u>
The derivative of any vector in an inertial reference frame can be written as



OXY = inertial reference frame
Oxy = reference frame attached to the body and rotating

The derivative of any vector $\vec{Q}$ is equal to $\left.\frac{d\vec{Q}}{dt}\right|_{OXY} = \left.\frac{d\vec{Q}}{dt}\right|_{Oxy} + \vec{\omega} \times \vec{Q}$. Therefore, to find the velocity of any point on the object we can define a position vector and differentiate.

Position vector to point A: $\vec{r}_{A/O}$

Velocity:

0 since the position between two point on a rigid body does not change

$$\vec{v}_A = \left.\frac{d\vec{r}_{A/O}}{dt}\right|_{OXY} = \left.\frac{d\vec{r}_{A/O}}{dt}\right|_{Oxy} + \vec{\omega}\times\vec{r}_{A/O} = \vec{\omega}\times\vec{r}_{A/O}$$

Acceleration:

$$\vec{a}_A = \left.\frac{d\vec{v}_A}{dt}\right|_{OXY} = \left.\frac{d\vec{v}_A}{dt}\right|_{Oxy} + \vec{\omega}\times\vec{v}_A = \vec{\alpha}\times\vec{r}_{A/O} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$

Note that the acceleration has two terms, one points in the tangential direction, (the $\vec{\alpha}\times\vec{r}_{A/O}$ term) and one points straight from the point back towards the origin, (the $\vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$ term).

Derivation using polar coordinates:

Recall from the discussion of polar coordinates

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

Now if we consider the point on a rigid body undergoing fixed axis rotation we know that r does not change magnitude so $\dot{r} = \ddot{r} = 0$ and that $\dot{\theta} = \omega,\ \ddot{\theta} = \alpha$ these equations reduce to

$$\vec{v} = r\omega\hat{e}_\theta = r\omega\hat{e}_t$$
$$\vec{a} = -r\omega^2\hat{e}_r + r\alpha\hat{e}_\theta = r\omega^2\hat{e}_n + r\alpha\hat{e}_t$$

which is identical to what we had before if we recognize that for fixed axis rotation (circular motion of every point) the unit vector in the transverse direction is identical to the tangential unit vector and the unit vector in the radial direction is equal to the negative of the unit vector in the normal direction.

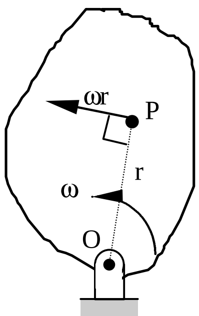
These equations can also be derived using the formulas for velocity and acceleration in normal and tangential coordinates.

Summary

1. There can be an acceleration component perpendicular to the radius vector with magnitude ra . We will call it the tangential acceleration.

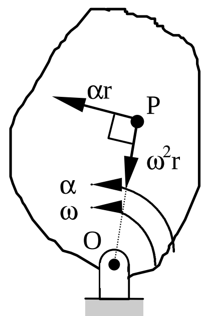
2. If there is an angular velocity present, there will always be an acceleration component with magnitude rw^2. The direction of this acceleration is always in exactly the opposite direction than the radius vector. This component is called the normal component.

Visual Summary: The velocity and acceleration of a point on a rigid body undergoing fixed axis rotation is shown below.



Velocity

$$\vec{v}_P = \vec{w} \times \vec{r}_{P/O}$$



Acceleration

$$\vec{a}_P = \vec{a} \times \vec{r}_{P/O} + \vec{w} \times \left(\vec{w} \times \vec{r}_{P/O}\right)$$
$$= \vec{a} \times \vec{r}_{P/O} - w^2 \vec{r}_{P/O} \quad \text{(plane motion)}$$

See examples 6.1 in text.

Example left for the student

A 1.25 m rod is pivoted at one end, and swings in the xy plane. At a certain instant, it is oriented at 20 degrees counterclockwise from the positive x-axis. It has an angular velocity of 0.6 rad/s and an angular acceleration of 0.75 rad/s^2, both clockwise. Find the velocity and acceleration of the free end of the rod.

(Ans: $\vec{v}$ = 0.75 rad/s ∠ 20° or $\vec{v} = -0.705\hat{i} - 0.256\hat{j}$ m/s

$\vec{a} = -1.035\hat{i} + 0.102\hat{j}$ m/s)