

TIME DECAY ESTIMATES FOR THE WAVE EQUATION WITH POTENTIAL IN DIMENSION TWO

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ABSTRACT. We study the wave equation with potential $u_{tt} - \Delta u + Vu = 0$ in two spatial dimensions, with V a real-valued, decaying potential. With $H = -\Delta + V$, we study a variety of mapping estimates of the solution operators, $\cos(t\sqrt{H})$ and $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$ under the assumption that zero is a regular point of the spectrum of H . We prove a dispersive estimate with a time decay rate of $|t|^{-\frac{1}{2}}$, a polynomially weighted dispersive estimate which attains a faster decay rate of $|t|^{-1}(\log |t|)^{-2}$ for $|t| > 2$. Finally, we prove dispersive estimates if zero is not a regular point of the spectrum of H .

1. INTRODUCTION

In this paper we study the linear wave equation with a real-valued decaying potential in two spatial dimensions,

$$(1) \quad u_{tt} - \Delta u + Vu = 0, \quad u(0) = f, \quad \partial_t u(0) = g, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Formally, with $H = -\Delta + V$, we represent the solution of this equation by

$$(2) \quad u(x, t) = \cos(t\sqrt{H})f(x) + \frac{\sin(t\sqrt{H})}{\sqrt{H}}g(x).$$

This formulation is valid if, for example, $f \in L^2$ and $g \in \dot{H}^{-1}$. In the free case, when $V = 0$, it is known that if the initial data f and g lie in sufficiently regular Sobolev spaces we also have the dispersive estimates

$$(3) \quad \|\cos(t\sqrt{-\Delta})f\|_\infty \lesssim |t|^{-1/2}\|f\|_{W^{k+1,1}},$$

$$(4) \quad \left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g \right\|_\infty \lesssim |t|^{-1/2}\|g\|_{W^{k,1}},$$

for $k > \frac{1}{2}$ in dimension two, see [26, 3, 4]. In general, one must use Hardy or Besov spaces or BMO to obtain the sharp $k = \frac{n-1}{2}$ smoothness bound in even dimensions. One can attain the bound for Sobolev spaces in dimension three where one can use the divergence theorem, see [35]. More generally, in odd spatial dimensions one can take advantage of the strong Huygens principle.

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In this paper, we extend these dispersive bounds to the perturbed equation, (1), when the initial data (f, g) is sufficiently differentiable with relatively weak assumptions on the potential V . These, along with the H^s bounds yield a class of L^p dispersive bounds, see Theorem 1.1 of [3].

Such dispersive estimates were first studied by Beals and Strauss in dimensions $n \geq 3$ in [4]. In two dimensions there is not much work on the $W^{k,1} \rightarrow L^\infty$ dispersive estimates or ‘regularized’ $L^1 \rightarrow L^\infty$ type estimates, where negative powers of H are employed. High frequency estimates of this type were studied by Moulin in [27]. In [23], Kopylova studied two dimensional local estimates, based on polynomially weighted H^s spaces when zero energy is regular, in which a decay rate of $t^{-1}(\log t)^{-2}$ is attained for large t . In the H^s setting, one can take advantage of conservation of energy, which is unavailable for the global dispersive bounds. This matches the decay rate obtained by Murata for the two dimensional Schrödinger equation in [28]. Dispersive bounds for the wave equation in three dimensions have been studied, see [24, 5] for example. More is known about Strichartz estimates for the wave equation, particularly in dimensions $n \geq 3$ see for example [17, 6, 7, 16, 11, 5].

When we perturb the free wave equation with a potential, one needs to project away from the eigenspaces. If we assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 2$, it is well known, see [30], that the spectrum of H is made of finitely many non-negative eigenvalues and the absolutely continuous spectrum $[0, \infty)$. The negative eigenvalues cause exponential growth of the solutions due to the action of the operators $\cos(t\sqrt{E})$ and $E^{-\frac{1}{2}} \sin(t\sqrt{E})$ with $\sqrt{E} \in i\mathbb{R}$. Accordingly, we first project away from the eigenvalues with P_{ac} , the projection onto the absolutely continuous spectrum.

Throughout the paper we use the notation $a- := a - \epsilon$ and $a+ := a + \epsilon$ for some small, but fixed, $\epsilon > 0$. Our first main result is the dispersive bound

Theorem 1.1. *If $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 3$ and if zero is a regular point of the spectrum of $H = -\Delta + V$, then*

$$\begin{aligned} \|\cos(t\sqrt{H})\langle H \rangle^{-\frac{3}{4}-} P_{ac}(H)f\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}, \\ \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} \langle H \rangle^{-\frac{1}{4}-} P_{ac}(H)g \right\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}. \end{aligned}$$

In addition, we prove a weighted version of this Theorem that attains a faster time decay rate.

Theorem 1.2. *If $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 3$ and if zero is a regular point of the spectrum of $H = -\Delta + V$, then for $t > 2$*

$$\begin{aligned} \|\cos(t\sqrt{H})\langle H \rangle^{-\frac{3}{4}-} P_{ac}(H)f\|_{L^{1, \frac{1}{2}+} \rightarrow L^{\infty, -\frac{1}{2}-}} &\lesssim t^{-1}(\log t)^{-2}, \\ \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} \langle H \rangle^{-\frac{1}{4}-} P_{ac}(H)g \right\|_{L^{1, \frac{1}{2}+} \rightarrow L^{\infty, -\frac{1}{2}-}} &\lesssim t^{-1}(\log t)^{-2}. \end{aligned}$$

Here $L^{1, \frac{1}{2}+} := \{f : \mathbb{R}^2 \rightarrow \mathbb{C} : \|f\langle \cdot \rangle^{\frac{1}{2}+}\|_1 < \infty\}$ and $L^{\infty, -\frac{1}{2}-}$ is defined similarly. We note that the bounds in Theorems 1.1 and 1.2 can be replaced by bounds without negative powers of $\langle H \rangle$ which map $W^{2,1} \rightarrow L^\infty$ for the cosine operator and $W^{1,1} \rightarrow L^\infty$ for the sine operator (and respective

weighted Sobolev spaces for Theorem 1.2), see Remark 3.5 below. We note that the regularizing powers of $\langle H \rangle^{-\alpha}$ for $\alpha > 0$ reflect the loss of derivatives of the initial data, and are only needed for high energy. In the low energy estimates, we prove $L^1 \rightarrow L^\infty$ bounds.

A series of papers have considered such estimates involving ‘regularizing’ powers of H , mostly with a high-energy cut-off and using semi-classical techniques. See [9, 27] for two dimensional results. In higher dimensions see, for example, [36, 10].

In the case that zero is not regular, that is there are non-trivial bounded solutions to $H\psi = 0$, one can study the boundedness of sine and cosine operators. The distributional solutions of this equation have been characterized, see [20, 13], in terms of the L^p spaces. If $\psi \in L^\infty$ and $\psi \notin L^p$ for any $p < \infty$ we call ψ an ‘s-wave’ resonance. If $\psi \in L^p$ for all $p > 2$ but not in L^2 , we say ψ is a ‘p-wave’ resonance. Whereas, if $\psi \in L^2$ there is an eigenvalue at zero. It is known that when zero is not regular, in general, time decay is lost in the dispersive estimates. See, for example, [19, 15] for three-dimensional and [13] for two-dimensional Schrödinger operators. In the two-dimensional Schrödinger equation, it is shown that the ‘s-wave’ resonance is sufficiently mild as to not destroy the dispersive time decay rate. We prove a similar characterization for the wave equation.

Theorem 1.3. *Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$. If there is only an s-wave resonance at zero energy, then for $\beta > 4$, for $|t| > 1$ we have*

$$\begin{aligned} \|\cos(t\sqrt{H})\langle H \rangle^{-\frac{3}{4}} P_{ac}(H)f\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}, \\ \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} \langle H \rangle^{-\frac{1}{4}} P_{ac}(H)g \right\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}. \end{aligned}$$

If there is a p-wave resonance or eigenvalue at zero, then for $\beta > 6$, there are time-dependent operators F_t and G_t such that for $|t| > 1$,

$$\begin{aligned} \|\cos(t\sqrt{H})[\langle H \rangle^{-\frac{3}{4}} P_{ac}(H) - F_t]f\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}, \\ \left\| \left[\frac{\sin(t\sqrt{H})}{\sqrt{H}} \langle H \rangle^{-\frac{1}{4}} P_{ac}(H) - G_t \right] g \right\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}. \end{aligned}$$

with

$$\sup_t \|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1, \quad \text{and} \quad \|G_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|.$$

Estimates for the three-dimensional wave equation when there is a resonance at zero energy were obtained by Krieger and Schlag in [24], and (with Nakanishi) extended to mixed space-time estimates in [25]. In that work the authors were concerned with the focusing, energy critical non-linear wave equation in three spatial dimensions, in which the dispersive evolution (after projecting away from the resonance function) was needed. During the review period for this article, Erdoğan, Goldberg the author proved dispersive estimates for the four-dimensional Schrödinger and wave equations with resonances at zero energy, [12].

The resolvent operator $R_V(z) := (-\Delta + V - z)^{-1}$, is defined for $z \in \mathbb{C}$ which are not in the spectrum of H . We define the limiting operators

$$R_V^\pm(\lambda^2) := \lim_{\epsilon \rightarrow 0^+} (-\Delta + V - (\lambda^2 \pm i\epsilon))^{-1}.$$

Such operators are well defined on certain weighted L^2 spaces due to Agmon's limiting absorption principle, see [2].

Using the Spectral theorem we can write

$$\langle f(H)P_{ac}g, h \rangle = \int_0^\infty f(\lambda) \langle E'_{ac}(\lambda)g, h \rangle d\lambda,$$

where the Stone formula yields that the absolutely continuous spectral measure is given by

$$\langle E'_{ac}(\lambda)g, h \rangle = \frac{1}{2\pi i} \langle [R_V^+(\lambda) - R_V^-(\lambda)]g, h \rangle.$$

The key insight here is that the absolutely continuous spectral measure is the same as in the analysis of the Schrödinger equation with potential. The fact that the spectral measure is the difference of the resolvent operators is crucial to establishing dispersive bounds in the free case, and for low energy (small λ) bounds we establish in Section 4. For low energy, the difference of the resolvents helps to control the singularities of the resolvents as the spectral parameter $\lambda \rightarrow 0$. We write the evolution of the solution operator projection onto the absolutely continuous spectrum as

$$(5) \quad \cos(t\sqrt{H})P_{ac}f(x) + \frac{\sin(t\sqrt{H})}{\sqrt{H}}P_{ac}g(x) = \frac{1}{\pi i} \int_0^\infty \lambda \cos(t\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f(x) d\lambda \\ + \frac{1}{\pi i} \int_0^\infty \sin(t\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)]g(x) d\lambda.$$

Where we made the change of variables $\lambda \mapsto \lambda^2$ for computational convenience. The solution operators in the wave equation lead to multiplication in the spectral parameter by the functions sine and cosine instead of multiplication by $e^{it\lambda^2}$ that arises in the Schrödinger evolution. Throughout this paper, we will consider the case of $t > 0$, the case of $t < 0$ follows from minor modifications of our proofs.

The dispersive decay, in the sense of mappings between weighted L^2 spaces or from L^1 to L^∞ is well studied for the perturbed Schrödinger equation

$$i\partial_t u - \Delta u + Vu = 0, \quad u(x, 0) = f(x).$$

In two spatial dimensions, Murata attained time-integrable estimates on weighted L^2 spaces. This time-decay rate was attained by Erdoğan and the author in a logarithmically weighted L^1 to logarithmically weighted L^∞ space in [14]. This paper proves theorems for the wave equation which are analogous to those found in the work of Schlag in [31] and previous work of Erdoğan and the author, [13, 14] studying the Schrödinger solution operator $e^{itH}P_{ac}$ in two spatial dimensions.

The paper is organized as follows: In Section 2 we develop expansions for the two-dimensional free resolvent needed for our analysis. In Section 3 we establish the high-energy dispersive bound for Theorem 1.1. In Section 4 we recall necessary resolvent expansions for small λ and establish the

dispersive bound of Theorem 1.1 for low energy. In Section 5 we establish refinements of the bounds and expansions of the previous sections needed to prove Theorem 1.2. Finally in Section 6 we show how the low-energy evolution is affected by the presence of an eigenvalue and/or resonance at zero energy, and prove Theorem 1.3.

2. THE FREE RESOLVENT

In this section we discuss the properties of the free resolvent, $R_0^\pm(\lambda^2) = [-\Delta - (\lambda^2 \pm i0)]^{-1}$, in \mathbb{R}^2 . We wish to understand the free resolvent in order to better understand the spectral measure in (5). A similar discussion appears in [31, 13, 14].

To simplify the formulas, we use the notation

$$f = \tilde{O}(g)$$

to denote

$$\frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, 3, \dots$$

If the derivative bounds hold only for the first k derivatives we write $f = \tilde{O}_k(g)$.

Recall that

$$(6) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} H_0^\pm(\lambda|x-y|) = \pm \frac{i}{4} J_0(\lambda|x-y|) - \frac{1}{4} Y_0(\lambda|x-y|).$$

Thus, we have

$$(7) \quad R_0^+(\lambda^2)(x, y) - R_0^-(\lambda^2)(x, y) = \frac{i}{2} J_0(\lambda|x-y|).$$

From the series expansions for the Bessel functions, see [1], we have, as $z \rightarrow 0$,

$$(8) \quad J_0(z) = 1 - \frac{1}{4}z^2 + \tilde{O}_4(z^4),$$

$$(9) \quad \begin{aligned} Y_0(z) &= \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi}\left(\frac{1}{4}z^2 + \tilde{O}_4(z^4)\right) \\ &= \frac{2}{\pi}\log(z/2) + \frac{2\gamma}{\pi} + \tilde{O}(z^2 \log(z)). \end{aligned}$$

For $|z| > 1$,

$$(10) \quad H_0^\pm(z) = e^{\pm iz}\omega_\pm(z), \quad \omega_\pm(z) = \tilde{O}((1+|z|)^{-\frac{1}{2}}).$$

So that for $|z| > 1$

$$(11) \quad \mathcal{C}(z) = e^{iz}\omega_+(z) + e^{-iz}\omega_-(z), \quad \omega_\pm(z) = \tilde{O}((1+|z|)^{-\frac{1}{2}}),$$

for any $\mathcal{C} \in \{J_0, Y_0\}$ respectively with different ω_\pm that satisfy the same bounds.

In particular,

$$(12) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \log(\lambda|x-y|/2) + \tilde{O}(\lambda^2|x-y|^2 \log(\lambda|x-y|)), \quad \lambda|x-y| \ll 1.$$

$$(13) \quad R_0^\pm(\lambda^2)(x, y) = e^{\pm i\lambda|x-y|}\omega_\pm(\lambda|x-y|), \quad \lambda|x-y| \gtrsim 1.$$

3. HIGH ENERGY

Here we show high energy, $\lambda \geq \lambda_1 > 0$ for some fixed $\lambda_1 > 0$, estimates for the evolution of the solution of (1) in (5). The exact value of $\lambda_1 > 0$ is unimportant for the following analysis, it will be determined by certain resolvent expansions used in the low energy regime, see Lemma 4.6 below.

Take a smooth cut-off function $\chi \in C^\infty((0, \infty))$ with $\chi(x) = 1$ if $x < \frac{1}{2}$ and $\chi(x) = 0$ if $x > 1$. Define $\tilde{\chi} = 1 - \chi$, and $\chi_1(x) = \chi(x/\lambda_1)$ with $\tilde{\chi}_1 = 1 - \chi_1$. For the high energy evolution, we prove the following high energy version of Theorem 1.1.

Proposition 3.1. *For fixed $\lambda_1 > 0$, if $|V(x)| \lesssim \langle x \rangle^{-2-}$, we have the bounds*

$$\begin{aligned} \|\cos(t\sqrt{H})H^{-\frac{3}{4}-}\tilde{\chi}_1(\sqrt{H})\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}, \\ \|\sin(t\sqrt{H})H^{-\frac{3}{4}-}\tilde{\chi}_1(\sqrt{H})\|_{L^1 \rightarrow L^\infty} &\lesssim |t|^{-\frac{1}{2}}. \end{aligned}$$

We define the cut-off $\chi_L(\lambda) = \chi(\lambda/L)$. Due to the spectral representation, (5), the Proposition follows if we prove the bound

$$(14) \quad \sup_{x, y \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_{\mathbb{R}^2} \int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) \chi_L(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim |t|^{-\frac{1}{2}}.$$

We prove the estimates with $e^{it\lambda}$, estimates for $e^{-it\lambda}$ and hence $\sin(t\lambda)$ and $\cos(t\lambda)$ follow similarly.

We note that such a bound is attained in [27] for λ_1 large enough with the assumption that

$$\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|V(z)|}{|z - y|^{\frac{1}{2}}} dz \leq C < \infty.$$

As one needs more decay on the potential to understand the low energy evolution and we need the bound to hold for any $\lambda_1 > 0$, we provide the following proof which requires a slightly stronger assumption on the potential.

We start with the resolvent expansion

$$(15) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2) V R_0^\pm(\lambda^2) + R_0^\pm(\lambda^2) V R_V^\pm(\lambda^2) V R_0^\pm(\lambda^2).$$

The first term is the free resolvent. In the high energy regime for the first term only, we need to use the cancellation between ‘+’ and ‘-’ free resolvents, for the remaining terms the cancellation between ‘+’ and ‘-’ resolvents does not simplify the proofs and is not used.

Lemma 3.2. *We have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) \chi_L(\lambda) [R_0^+(\lambda^2) - R_0^-(\lambda^2)](x, y) d\lambda \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. Using (7), we reduce the contribution of the first term in (15) to showing the bound

$$\sup_{x, y \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} J_0(\lambda|x - y|) d\lambda \right| \lesssim t^{-\frac{1}{2}}.$$

Using (8) and (10) we write

$$J_0(\lambda|x - y|) = \rho_-(\lambda|x - y|) + e^{i\lambda|x - y|} \omega_+(\lambda|x - y|) + e^{-i\lambda|x - y|} \omega_-(\lambda|x - y|)$$

with $\rho_-(z) = \tilde{O}(1)$ supported on $[0, \frac{1}{2}]$ and $\omega_\pm(z) = \tilde{O}((1+z)^{-1/2})$ supported on $[\frac{1}{4}, \infty)$.

We first consider the contribution of J_0 when $\lambda|x-y| \gtrsim 1$ we consider first the ‘-’ phase from the expansion for J_0 and bound the following integral

$$(16) \quad \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda|x-y|} \omega_-(\lambda|x-y|) d\lambda.$$

First we note that if $t - |x-y| \leq t/2$ then $t \lesssim |x-y|$ and we use the decay of ω_- to see

$$|(16)| \lesssim \int_1^\infty \lambda^{-1-} |x-y|^{-\frac{1}{2}-} d\lambda \lesssim |x-y|^{-\frac{1}{2}-} \lesssim t^{-\frac{1}{2}-}.$$

On the other hand, if $t - |x-y| \geq t/2$, we can integrate by parts against the combined phase $e^{i\lambda(t-|x-y|)}$ to see

$$\begin{aligned} |(16)| &\lesssim \frac{1}{t-|x-y|} \int_0^\infty \frac{d}{d\lambda} (\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \omega_-(\lambda|x-y|)) d\lambda \lesssim \frac{1}{t} \int_1^\infty \frac{\lambda^{-\frac{3}{2}-}}{(1+\lambda|x-y|)^{\frac{1}{2}}} d\lambda \\ &\lesssim \frac{1}{t} |x-y|^{-\frac{1}{2}} \int_{\lambda \gtrsim |x-y|^{-1}} \lambda^{-2} d\lambda \lesssim \frac{|x-y|^{\frac{1}{2}}}{t} \lesssim t^{-\frac{1}{2}}. \end{aligned}$$

Where we used that $\lambda \gtrsim 1$ in the second to last step and that $|x-y| \lesssim t$ in the last inequality. For the ‘+’ phase, the analysis follows from considering the cases of $t + |x-y| \leq 2t$ and $t + |x-y| \geq 2t$ in place of the cases $t - |x-y| \geq t/2$ and $t - |x-y| \leq t/2$ considered for the negative phase.

For the contribution of ρ_- we need to consider three cases. First, consider the case when $t \lesssim |x-y|$. Then, using $1 \lesssim (\lambda|x-y|)^{-1}$ on the support of ρ_- we have

$$\int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \chi_{\{|x-y| \lesssim 1\}} d\lambda \lesssim \frac{1}{|x-y|^{\frac{1}{2}}} \int_0^\infty \lambda^{-1-} \tilde{\chi}_1(\lambda) d\lambda \lesssim \frac{1}{|x-y|^{\frac{1}{2}}} \lesssim t^{-\frac{1}{2}}.$$

We now consider the second case in which $t \gtrsim |x-y|$. The first subcase here is when $t \gtrsim 1$. We use the fact that $|\frac{d^k}{d\lambda^k} \tilde{\chi}_1(\lambda)|, |\frac{d^k}{d\lambda^k} \tilde{\chi}_L(\lambda)| \lesssim \lambda^{-k}$ to see

$$\begin{aligned} \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda &= \frac{1}{it} \int_0^\infty e^{it\lambda} \frac{d}{d\lambda} (\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1)) d\lambda \\ &\lesssim \frac{1}{t} \int_1^\infty \lambda^{-\frac{3}{2}-} d\lambda \lesssim t^{-1} \lesssim t^{-\frac{1}{2}}. \end{aligned}$$

Finally, we need to consider the case in which $|x-y| \ll t \ll 1$. Then we break the integral up into two pieces to bound with

$$\int_0^{t^{-1}} e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda + \int_{t^{-1}}^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda.$$

For the first integral we note that on the support of $\tilde{\chi}_1(\lambda)$, we have $\lambda^{-\frac{1}{2}-} \lesssim \lambda^{-\frac{1}{2}}$ so,

$$\left| \int_0^{t^{-1}} e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda \right| \lesssim \int_0^{t^{-1}} \lambda^{-\frac{1}{2}} d\lambda \lesssim t^{-\frac{1}{2}}$$

For the second integral, we integrate by parts once against the phase $e^{it\lambda}$ to bound with

$$\int_{t^{-1}}^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda \lesssim \frac{\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-}}{t} \Big|_{t^{-1}}^\infty + \frac{1}{t} \int_{t^{-1}}^\infty \lambda^{-\frac{3}{2}-} d\lambda \lesssim t^{-\frac{1}{2}}.$$

Again we use that that on the support of $\tilde{\chi}_1(\lambda)$, we have $\lambda^{-\frac{1}{2}-} \lesssim \lambda^{-\frac{1}{2}}$ to get the $t^{-\frac{1}{2}}$ bound. \square

It is worth noting one can obtain this high energy estimate by considering an alternative definition of the Bessel function to bound

$$\int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \left(\int_{|\omega|=1} e^{i\lambda\omega \cdot (x-y)} d\omega \right) d\lambda.$$

The desired bounds follow by using standard facts on the Fourier transforms of densities on curved manifolds, see [34] for example. We do not pursue this issue here, as much of our analysis requires considering the free resolvent and the Bessel functions on different regimes according to the size of their argument, recall (8), (9), (10), (12) and (13). This is due to the vastly different behavior of the ‘low’, $|z| \ll 1$ and ‘high’, $|z| \gtrsim 1$, portions of these functions.

For the second term in (15), we have

Lemma 3.3. *If $|V(x)| \lesssim \langle x \rangle^{-2-}$, then we have the bound*

$$\sup_{x,y \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_{\mathbb{R}^2} \int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) \chi_L(\lambda) R_0^\pm(\lambda^2)(x,z) V(z) R_0^\pm(\lambda^2)(z,y) d\lambda dz \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. We won’t make use of any cancellation between ‘ \pm ’ terms. Thus, we will only consider R_0^- , and drop the ‘ \pm ’ signs. Using (6), (8), (9), and (10) we write

$$(17) \quad R_0(\lambda^2)(x,y) = e^{-i\lambda|x-y|} \rho_+(\lambda|x-y|) + \rho_-(\lambda|x-y|),$$

where ρ_+ and ρ_- are supported on the sets $[1/4, \infty)$ and $[0, 1/2]$, respectively. Moreover, we have the bounds

$$(18) \quad \rho_-(y) = \tilde{O}(1 + |\log y|), \quad \rho_+(y) = \tilde{O}((1 + |y|)^{-1/2})$$

We now consider

$$\int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) \chi_L(\lambda) R_0^-(\lambda^2)(x,z) V(z) R_0^-(\lambda^2)(z,y) d\lambda.$$

We are considering the case of R_0^- as the sign of t and the high energy phases do not match, which requires a more delicate analysis. We note that

$$\begin{aligned} |\partial_\lambda^k [\rho_+(\lambda d_j)]| &\lesssim \frac{d_j^k}{(1 + \lambda d_j)^{k+1/2}}, \quad k = 0, 1, 2, \dots, \\ |\partial_\lambda^k [\rho_-(\lambda d_j)]| &\lesssim \frac{1}{\lambda^k}, \quad k = 1, 2, \dots \end{aligned}$$

Define the function $\log^-(z) = -(\log z) \chi_{\{0 < z < 1\}}$. Using the monotonicity of \log^- function, we also obtain

$$\tilde{\chi}_1(\lambda) |\rho_-(\lambda d_j)| \lesssim \tilde{\chi}_1(\lambda) (1 + |\log(\lambda d_j)|) \chi_{\{0 < \lambda d_j \leq 1/2\}} \lesssim \tilde{\chi}(\lambda) (1 + \log^-(\lambda d_j)) \lesssim 1 + \log^-(d_j).$$

Finally, noting that $(\tilde{\chi}_1)'$ and χ_L' are supported on the set $\{\lambda \approx 1\}$, we have

$$(19) \quad |\partial_\lambda^k \tilde{\chi}_1(\lambda)|, \quad |\partial_\lambda^k \chi_L(\lambda)| \lesssim \lambda^{-k}.$$

For notational convenience, let $d_1 := |x-z|$ and $d_2 := |z-y|$. First consider the ‘low-low’ interaction, that is when each resolvent contributes ρ_- . We consider two cases, first if $t \geq \sqrt{d_1 d_2}$, then

$$\begin{aligned} & \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2) d\lambda \\ & \lesssim \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} [\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2)] \right| d\lambda \\ & \lesssim \frac{(1 + \log^- d_1)(1 + \log^- d_2)}{t} \int_1^\infty \frac{1}{\lambda^{\frac{3}{2}+}} d\lambda \lesssim \frac{(1 + \log^- d_1)(1 + \log^- d_2)}{t^{\frac{1}{2}} d_1^{\frac{1}{4}} d_2^{\frac{1}{4}}}. \end{aligned}$$

On the other hand, if $t \leq \sqrt{d_1 d_2}$ we don’t integrate by parts and instead use the support conditions on ρ_- to see

$$(20) \quad \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2) d\lambda \lesssim \int_1^\infty \frac{\lambda^{-\frac{1}{2}-}}{(\lambda \sqrt{d_1 d_2})} d\lambda \lesssim \frac{1}{t^{\frac{1}{2}}} \frac{1}{d_1^{\frac{1}{4}} d_2^{\frac{1}{4}}}.$$

Next we consider the ‘low-high’ interaction. That is, we have a contribution from one ρ_+ and one ρ_- . Accordingly, we need to bound integrals of the form

$$(21) \quad \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda d_1} \rho_+(\lambda d_1) \rho_-(\lambda d_2) d\lambda$$

Due to the a-priori λ decay of ρ_+ , (10), we can take $L = \infty$ and remove the cut-off from our analysis. We need to consider two cases, depending on the relative size of t and the phase d_1 . In the first case, we consider $t - d_1 \geq \frac{t}{2}$. In this case we can safely integrate by parts against $e^{i\lambda(t-d_1)}$.

$$\begin{aligned} |(21)| & \lesssim \frac{1}{t-d_1} \int_0^\infty \left| \frac{d}{d\lambda} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_+(\lambda d_1) \rho_-(\lambda d_2)) \right| d\lambda \\ & \lesssim \frac{1}{t} \int_1^\infty \left[\frac{1 + \log^- d_2}{\lambda^{\frac{3}{2}+(1+\lambda d_1)^{\frac{1}{2}}}} + \frac{d_1(1 + \log^- d_2)}{\lambda^{\frac{1}{2}+(1+\lambda d_1)^{\frac{3}{2}}}} \right] d\lambda \\ & \lesssim \frac{1 + \log^- d_2}{t} \int_1^\infty \lambda^{-\frac{3}{2}-} d\lambda \lesssim \frac{1 + \log^- d_2}{t} \lesssim \frac{1 + \log^- d_2}{t^{\frac{1}{2}} d_1^{\frac{1}{2}}} \end{aligned}$$

Here we used that $t^{-1} \lesssim d_1^{-1}$ in the last inequality.

The second case is when $t - d_1 \leq \frac{t}{2}$. In this case we won’t integrate by parts, but use that $d_1^{-1} \lesssim t^{-1}$.

$$|(21)| \lesssim \int_1^\infty \frac{\lambda^{-\frac{1}{2}-}(1 + \log^- d_2)}{(1 + \lambda d_1)^{\frac{1}{2}}} d\lambda \lesssim \frac{1 + \log^- d_2}{d_1^{\frac{1}{2}}} \int_1^\infty \lambda^{-1-} d\lambda \lesssim \frac{1 + \log^- d_2}{t^{\frac{1}{2}}}.$$

Note that it is precisely this case that constrains us to a $t^{-\frac{1}{2}}$ time decay rate in the wave equation for a non-weighted L^∞ bound.

For the ‘high-high’ interaction, we need to bound

$$(22) \quad \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda(d_1+d_2)} \rho_+(\lambda d_1) \rho_+(\lambda d_2) d\lambda.$$

We first consider the case when $t - (d_1 + d_2) \leq \frac{t}{2}$, in which case $(d_1 + d_2) \geq \frac{t}{2}$. Then

$$|(22)| \lesssim \frac{1}{\sqrt{d_1 d_2}} \int_1^\infty \lambda^{-\frac{3}{2}-} d\lambda \lesssim \frac{1}{\sqrt{d_1 d_2}} \lesssim \frac{1}{t^{\frac{1}{2}} \sqrt{\min(d_1, d_2)}}$$

Where we used that $(d_1 + d_2) \geq \frac{t}{2}$ gives us that $\max(d_1, d_2) \gtrsim t$.

The second case is when $t - (d_1 + d_2) \geq \frac{t}{2}$, in which case we don’t integrate by parts once but use $d_1, d_2 \lesssim t$ and the asymptotics of ρ_+ . Accordingly we bound

$$\frac{1}{\sqrt{d_1 d_2}} \int_0^\infty \tilde{\chi}_1(\lambda) \lambda^{-\frac{3}{2}-} \rho_+(\lambda d_1) \rho_+(\lambda d_2) d\lambda \lesssim \frac{1}{\sqrt{d_1 d_2}} \int_1^\infty \lambda^{-\frac{3}{2}-} d\lambda \lesssim \frac{1}{t^{\frac{1}{2}} \sqrt{\min(d_1, d_2)}}$$

The case when R_0^+ appears and the exponentials have a positive phase follows as in the case analysis above by considering the size of $t + d_1$ or $t + d_1 + d_2$ compared to $2t$.

Finally, we close the argument by noting that

$$\sup_{x,y} \left| \int_{\mathbb{R}^2} V(z) \left(1 + \frac{(1 + \log^- |z - y|)(1 + \log^- |x - z|)}{|x - z|^{\frac{1}{4}} |z - y|^{\frac{1}{4}}} + \frac{1 + \log^- |z - y|}{|x - z|^{\frac{1}{2}}} \right) dz \right| \lesssim 1$$

provided that $|V(z)| \lesssim \langle z \rangle^{-2-}$.

□

To control the remainder of the born series we employ the limiting absorption principle of Agmon, [2]. For $\lambda > \lambda_1 > 0$

$$(23) \quad \|R_0^\pm(\lambda^2)\|_{L^{2,\sigma}(\mathbb{R}^2) \rightarrow L^{2,-\sigma}(\mathbb{R}^2)} \lesssim \lambda^{-1+}.$$

From the limiting absorption principle, we can deduce bounds for the derivatives by considering the representation for the free resolvent in (17). Specifically, we have

$$(24) \quad \|R_V^\pm(\lambda^2)\|_{L^{2,\sigma}(\mathbb{R}^2) \rightarrow L^{2,-\sigma}(\mathbb{R}^2)} \lesssim \lambda^{-1+},$$

$$(25) \quad \sup_{\lambda > \lambda_1} \|\partial_\lambda^k R_V^\pm(\lambda^2)\|_{L^{2,\sigma}(\mathbb{R}^2) \rightarrow L^{2,-\sigma}(\mathbb{R}^2)} \lesssim 1,$$

which is valid for $\sigma > \frac{1}{2} + k$. The bounds for the derivatives are valid for the free resolvents as well. The proof of these estimates follows as in Proposition 9 in [18] along with the discussion following it. The bounds for the perturbed resolvent relies on the resolvent identity

$$R_V^\pm(\lambda^2) = (I + R_0^\pm(\lambda^2)V)^{-1} R_0^\pm(\lambda^2)$$

and the absence of embedded eigenvalues in the continuous spectrum, which is guaranteed by Kato’s Theorem, see Section XIII.8 of [30].

Using the representation (18), we note the following bounds on the free resolvent which are valid on $\lambda > \lambda_1 > 0$,

$$(26) \quad |\partial_\lambda^k R_0^\pm(\lambda^2)(x, y)| \lesssim |x - y|^k \begin{cases} |\log(\lambda|x - y|)| & 0 < \lambda|x - y| < \frac{1}{2} \\ (\lambda|x - y|)^{-\frac{1}{2}} & \lambda|x - y| \gtrsim 1 \end{cases} \lesssim \lambda^{-\frac{1}{2}} |x - y|^{k - \frac{1}{2}}.$$

Thus, for $\sigma > \frac{1}{2} + k$,

$$(27) \quad \|\partial_\lambda^k R_0^\pm(\lambda^2)(x, y) \langle y \rangle^{-\sigma}\|_{L_y^2} \lesssim \lambda^{-\frac{1}{2}} \left[\int_{\mathbb{R}^2} \frac{|x - y|^{2k-1}}{\langle y \rangle^{2\sigma}} dy \right]^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \langle x \rangle^{\max(0, k - \frac{1}{2})}.$$

To avoid polynomial weights in x or y , we take a bit more care with the leading and lagging free resolvents. We use the following estimates on the pieces of the free resolvent in (17). Noting that for $\lambda \gtrsim 1$,

$$(28) \quad |\partial_\lambda^k \rho_-(\lambda)(\lambda r)| \lesssim (1 + \lambda r)^{-\frac{1}{2}} \begin{cases} (1 + \log^- r) & k = 0 \\ \lambda^{-1} & k = 1 \end{cases}$$

$$(29) \quad |\partial_\lambda^k \rho_+(\lambda)(\lambda r)| \lesssim \lambda^{-k} (1 + \lambda r)^{-\frac{1}{2}}, \quad k = 0, 1.$$

The $(1 + \lambda r)^{-\frac{1}{2}}$ in the first bound follows from the support conditions on ρ_- . So that for $\sigma > k + \frac{1}{2}$ and $k = 0, 1$ we have the weighted L^2 bounds

$$(30) \quad \|\partial_\lambda^k \rho_\pm(\lambda|x - y|)\|_{L_y^{2, \sigma}} \lesssim \lambda^{-k - \frac{1}{2}}.$$

Once again, we estimate the R_V^+ and R_V^- terms separately and omit the ‘ \pm ’ signs.

Lemma 3.4. *If $|V(x)| \lesssim \langle x \rangle^{-2-}$ we have the bound (14) as*

$$(31) \quad \sup_{x, y \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \mathcal{E}(\lambda)(x, y) d\lambda \right| \lesssim t^{-\frac{1}{2}}$$

where

$$\mathcal{E}(\lambda)(x, y) = \tilde{\chi}_1(\lambda) \chi_L(\lambda) \langle V R_V^\pm(\lambda^2) V R_0^\pm(\lambda^2)(\cdot, x), R_0^\pm(\lambda^2)(\cdot, y) \rangle.$$

Proof. Using (27), one can see that

$$\begin{aligned} |(31)| &\lesssim \int_0^\infty \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \|R_0\|_{L^{2, -\frac{1}{2}-}} \|V\|_{L^{2, -\frac{1}{2}-} \rightarrow L^{2, \frac{1}{2}+}} \|R_V\|_{L^{2, \frac{1}{2}+} \rightarrow L^{2, -\frac{1}{2}-}} \\ &\quad \times \|V\|_{L^{2, -\frac{1}{2}-} \rightarrow L^{2, \frac{1}{2}+}} \|R_0\|_{L^{2, -\frac{1}{2}-}} d\lambda \lesssim \int_0^\infty \tilde{\chi}_1(\lambda) \lambda^{-\frac{3}{2}-} d\lambda \lesssim 1. \end{aligned}$$

Again this justifies taking $L = \infty$ in the cut-off χ_L . To establish bounds with time decay, we again need to distinguish cases based on whether the leading and lagging resolvents contribute ρ_- or ρ_+ and the size of t relative to $|x - z|$, and/or $|y - z|$. The analysis is essentially identical to the second term of the Born Series, but here we use weighted L^2 bounds. For example, we can conclude the bound for the term analogous to (22) by noting that

$$\|1 + \log^- |x - \cdot|\|_{L^{2, -1-}} \|V\|_{L^{2, -1-} \rightarrow L^{2, \frac{1}{2}+}} \|R_V^\pm\|_{L^{2, \frac{1}{2}+} \rightarrow L^{2, -\frac{1}{2}-}} \|V\|_{L^{2, -\frac{1}{2}-} \rightarrow L^{2, \frac{1}{2}+}} \| |y - \cdot|^{-\frac{1}{2}} \|_{L^{2, -\frac{1}{2}-}}$$

is bounded uniformly in x and y under the assumptions on the potential.

We need only make a few small adjustments. We note that, when integrating by parts, if the derivative acts on the resolvent $R_V^\pm(\lambda^2)$ in the case when only ρ_- 's are contributing we must account for the case when the derivative acts on the perturbed resolvent,

$$\begin{aligned} \int_0^\infty \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2) \partial_\lambda R_V^\pm(\lambda^2) d\lambda \\ \lesssim \frac{1}{d_1^{\frac{1}{4}} d_2^{\frac{1}{4}}} \int_0^\infty \tilde{\chi}_1(\lambda) \lambda^{-1-} \rho_-(\lambda d_1) \rho_-(\lambda d_2) \partial_\lambda R_V^\pm(\lambda^2) d\lambda \end{aligned}$$

by using the support conditions that $\lambda d_j \lesssim 1$. Then, using (26) one can see $\|r^{-\frac{1}{4}} \rho_-(\lambda r)\|_{L^{2,\sigma}} \lesssim 1$ for $\sigma > \frac{1}{2}$.

This argument requires that $|V(x)| \lesssim \langle x \rangle^{-2-}$. One can see that the requirement on the decay rate of the potential arises when, for instance, the λ derivative act on one of the ρ_\pm 's, this differentiated object is in $L^{2,-\frac{3}{2}-}$ by (30). The potential then needs to map $L^{2,-\frac{3}{2}-} \rightarrow L^{2,\frac{1}{2}+}$ for the application of the limiting absorption principle for R_V . On the other hand, if the derivative acts on the perturbed resolvent, $\partial_\lambda R_V$ maps $L^{2,\frac{3}{2}+} \rightarrow L^{2,-\frac{3}{2}-}$ by (24), and V must map $L^{2,-\frac{3}{2}-}$ to $L^{2,\frac{1}{2}+}$. \square

We can now prove the high energy bound.

Proof of Proposition 3.1. The statement follows from the spectral representation (14), and Lemmas 3.3 and 3.4 to control each term in the Born series expansion (15). \square

Remark 3.5. *We note that one can prove estimates without the ‘regularizing’ powers of H by directly appealing to differentiability of the initial data. That is, one can show*

$$\begin{aligned} \sup_{x \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_{\mathbb{R}^2} \int_0^\infty e^{it\lambda} \lambda \tilde{\chi}_1(\lambda) \chi_L(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) f(y) d\lambda dy \right| \lesssim |t|^{-\frac{1}{2}} \|f\|_{W^{2,1}} \\ \sup_{x \in \mathbb{R}^2} \sup_{L \geq 1} \left| \int_{\mathbb{R}^2} \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) f(y) d\lambda dy \right| \lesssim |t|^{-\frac{1}{2}} \|f\|_{W^{1,1}} \end{aligned}$$

To do this, one can employ an integration by parts in the y spatial variable according to

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-i\lambda|z-y|} \rho_+(\lambda|z-y|) f(y) dy &= \frac{i}{\lambda} \int_{\mathbb{R}^2} e^{-i\lambda|z-y|} \nabla_y \cdot \left[\rho_+(\lambda|z-y|) f(y) \frac{y-z}{|y-z|} \right] dy \\ &= -\frac{1}{\lambda^2} \int_{\mathbb{R}^2} e^{-i\lambda|z-y|} \nabla_y \cdot \left\{ \nabla_y \cdot \left[\rho_+(\lambda|z-y|) f(y) \frac{y-z}{|y-z|} \right] \frac{y-z}{|y-z|} \right\} dy. \end{aligned}$$

to gain λ decay to ensure integrability as $\lambda \rightarrow \infty$ for the ‘high-high’ interactions. One must also iterate the resolvent identity to form a longer Born series expansion than used in (15), and use similar (but modified) techniques as we used above. This method, however, misses the sharp smoothness requirement by $\frac{1}{2}$ of a derivative instead of the ϵ loss presented in this section.

4. LOW ENERGY

To analyze the evolution of (2) on low energy, we must utilize different expansions for the resolvent. The low energy contribution to Theorem 1.1 can again be reduced to an integral bound due to (5).

Proposition 4.1. *If $|V(x)| \lesssim \langle x \rangle^{-3-}$ and if zero is a regular point of the spectrum of $H = -\Delta + V$, we have the bound*

$$\sup_{x,y \in \mathbb{R}^2} \left| \int_0^\infty (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) [R_V^+(x,y) - R_V^-(x,y)] d\lambda \right| \lesssim t^{-\frac{1}{2}}.$$

First we state the needed resolvent expansions and then control their contribution before proving the dispersive bound.

4.1. Low energy Resolvent expansions. Let $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, and let $v = |V|^{1/2}$. We have $V = Uv^2$. We use the symmetric resolvent identity, valid for $\Im \lambda > 0$:

$$(32) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2),$$

where $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$. The key issue studied in [20] and used in [31, 13, 14] in the resolvent expansions is the invertibility of the operator $M^\pm(\lambda)$ for small λ under various spectral assumptions at zero. For the sake of brevity, we use the expansions of [31, 13, 14] as needed and omit the proofs. To understand the operator $M^\pm(\lambda)$ we define the following (see (9)),

$$(33) \quad G_0 f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy = (-\Delta)^{-1} f(x),$$

$$(34) \quad g^\pm(\lambda) := \|V\|_1 \left(\pm \frac{i}{4} - \frac{1}{2\pi} \log(\lambda/2) - \frac{\gamma}{2\pi} \right).$$

We consider more complicated expansions for $M^\pm(\lambda)$ in Sections 5 and 6.

Our first expansion is a modification of Lemma 5 in [31] which is Lemma 1 in [14].

Lemma 4.2. *We have the following expansion for the kernel of the free resolvent*

$$R_0^\pm(\lambda^2)(x,y) = \frac{1}{\|V\|_1} g^\pm(\lambda) + G_0(x,y) + E_0^\pm(\lambda)(x,y).$$

Here $G_0(x,y)$ is the kernel of the operator G_0 in (33), $g^\pm(\lambda)$ is as in (34), and E_0^\pm satisfies the bounds

$$|E_0^\pm| \lesssim \lambda^{\frac{1}{2}} |x-y|^{\frac{1}{2}}, \quad |\partial_\lambda E_0^\pm| \lesssim \lambda^{-\frac{1}{2}} |x-y|^{\frac{1}{2}}, \quad |\partial_\lambda^2 E_0^\pm| \lesssim \lambda^{-\frac{3}{2}} |x-y|^{\frac{3}{2}}.$$

We note that the expansion for the free resolvent we develop here appears to depend on the potential V through the factor of $\|V\|_1$ that appears in the functions $g^\pm(\lambda)$. We include this factor here to simplify certain operators later on and note that our goal is to develop an expansion for the operators $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$ which do explicitly depend on the potential.

This expansion can be interpreted as follows, the first term $g^\pm(\lambda)$ contains all of the singularities in λ , the second term is the integral kernel of the Green's function for $-\Delta$ and the remaining term

is an error term which is small in λ . Unlike in higher dimensional cases, we cannot simply expand the free resolvent as a small λ perturbation of the integral kernel operator $-(\Delta)^{-1}(x, y)$ due to the logarithmic behavior of $g^\pm(\lambda)$.

We employ this notation to match with that of previous works, see [31, 13, 14], in the context of the Schrödinger equation. The error terms in the expansions are not related to and should not be confused with the spectral family E'_{ac} in the spectral theorem. We note that the bound on the second derivative of the error term is not used here, but is needed to consider weighted estimates in Section 5.

Lemma 4.3. *For $\lambda > 0$ define $M^\pm(\lambda) := U + vR_0^\pm(\lambda^2)v$. Let $P = v\langle \cdot, v \rangle \|V\|_1^{-1}$ denote the orthogonal projection onto v . Then*

$$M^\pm(\lambda) = g^\pm(\lambda)P + T + E_1^\pm(\lambda).$$

Here $T = U + vG_0v$ where G_0 is an integral operator defined in (33). Further, the error term satisfies the bound

$$\left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E_1^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \lesssim 1,$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-}$.

Proof. This follows from Lemma 4.2 the definition of the operators

$$M^\pm(\lambda)^{-1} = U + vR_0^\pm(\lambda^2)v$$

□

We recall the following definition from [31] and [13].

Definition 4.4. *We say an operator $T : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

It is worth noting that finite rank operators and Hilbert-Schmidt operators are absolutely bounded. Also recall the following definition from [20], also see [31, 13].

Definition 4.5. *Let $Q := \mathbb{1} - P$. We say zero is a regular point of the spectrum of $H = -\Delta + V$ provided $QTQ = Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.*

(See Definition 6.1 in Section 6 below for a more complete description of regularity at zero.) In Lemma 8 of [31], it was proved that if zero is regular, then the operator $D_0 := (QTQ)^{-1}$ is absolutely bounded on QL^2 . Below, we discuss the invertibility of $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$, for small λ . This following lemma was proved in Lemma 8 of [31] by employing an abstract Feshbach inversion formula.

Lemma 4.6. *Suppose that zero is a regular point of the spectrum of $H = -\Delta + V$. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda)$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^2)$. Further, one has*

$$(35) \quad M^\pm(\lambda)^{-1} = h_\pm(\lambda)^{-1}S + QD_0Q + E^\pm(\lambda),$$

Here $h_\pm(\lambda) = g^\pm(\lambda) + c$ (with $c \in \mathbb{R}$), and

$$(36) \quad S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix}$$

is a finite-rank operator with real-valued kernel. Further, the error term satisfies the bounds

$$\left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \right\|_{HS} \lesssim 1,$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-}$.

Remark. Under the conditions of Theorem 1.1, the resolvent identity

$$(37) \quad \begin{aligned} R_V^\pm(\lambda^2) &= R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2) \\ &= R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)\frac{vSv}{h_\pm(\lambda)}R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vQD_0QvR_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vE^\pm(\lambda)vR_0^\pm(\lambda^2) \end{aligned}$$

holds as an operator identity between the spaces $L^{2, \frac{1}{2}+}(\mathbb{R}^2)$ and $L^{2, -\frac{1}{2}-}(\mathbb{R}^2)$, as in the limiting absorption principle, [2].

4.2. The low energy dispersive bound. We now consider the evolution for low energy, (5), when $0 < \lambda < \lambda_1$. In contrast with the high energy treatment in Section 3 we do not need any differentiability of initial data nor any ‘regularizing’ powers of H . In fact, for small λ , we prove an $L^1 \rightarrow L^\infty$ dispersive bound. We use previous work on dispersive bounds for the two-dimensional Schrödinger equation, using results of [31, 13, 14] as needed.

First we bound the free resolvent term contribution to (37),

Lemma 4.7. *We have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \left| \int_0^\infty e^{it\lambda} \chi_1(\lambda) [R_0^+(\lambda)(x, y) - R_0^-(\lambda)(x, y)] d\lambda \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. Since $R_0^+(\lambda)(x, y) - R_0^-(\lambda)(x, y) = \frac{i}{2}J_0(\lambda|x-y|)$. We consider the contribution of J_0 when $\lambda|x-y| \lesssim 1$, in which case $J_0(\lambda|x-y|) = 1 + \tilde{O}(\lambda^2|x-y|^2)$, see (8). In this case we note that the boundedness of the integral is clear, to obtain the desired $t^{-1/2}$ bound we interpolate with

$$\left| \int_0^\infty e^{it\lambda} \chi_1(\lambda) (1 + |x-y|^2 \tilde{O}(\lambda^2)) d\lambda \right| \lesssim \frac{1}{t} \int_0^\infty \chi_1(\lambda) + \lambda|x-y|^2 \chi_{\{\lambda \lesssim |x-y|^{-1}\}} d\lambda \lesssim t^{-1}.$$

For the piece supported on $\lambda|x-y| \gtrsim 1$, we note that for the negative phase piece we have to bound

$$\int_0^\infty e^{it\lambda} \chi_1(\lambda) e^{-i\lambda|x-y|} \omega_-(\lambda|x-y|) d\lambda.$$

As $\omega_- \lesssim 1$, the integral is clearly bounded. To attain time decay, we again split into cases. First, if $t - |x - y| \geq \frac{t}{2}$ (or in the case of the ‘+’ phase) we integrate by parts once to bound with

$$\frac{1}{t} \int_{\lambda \gtrsim |x-y|^{-1}} \frac{|x-y|^{-\frac{1}{2}}}{\lambda^{\frac{3}{2}}} d\lambda \lesssim t^{-1}.$$

On the other hand if $t - |x - y| \leq \frac{t}{2}$ we note that $|x - y| \gtrsim t$ and we can bound this contribution by

$$\frac{1}{|x-y|^{\frac{1}{2}}} \int_0^1 \lambda^{-\frac{1}{2}} d\lambda \lesssim t^{-\frac{1}{2}}$$

as desired. \square

We now consider the contribution of the operator QD_0Q in (37).

Proposition 4.8. *We have the bound*

$$(38) \quad \sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda} \chi_1(\lambda) (1 + \lambda) v(x_1) QD_0Q(x_1, y_1) v(y_1) \right. \\ \left. \left[R_0^+(\lambda^2)(x, x_1) R_0^+(\lambda^2)(y_1, y) - R_0^-(\lambda^2)(x, x_1) R_0^-(\lambda^2)(y_1, y) \right] d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. We can reduce the contribution to that when there is a single J_0 and a single Y_0 due to the difference of the ‘+’ and ‘-’ terms in (5) and

$$R_0^+(\lambda^2)(x, x_1) R_0^+(\lambda^2)(y, y_1) - R_0^-(\lambda^2)(x, x_1) R_0^-(\lambda^2)(y, y_1) \\ = -\frac{i}{8} \left(Y_0(\lambda|x - x_1|) J_0(\lambda|y - y_1|) + J_0(\lambda|x - x_1|) Y_0(\lambda|y - y_1|) \right).$$

This is a beneficial whenever at least one of the Bessel functions is supported on ‘low’ energy. For ‘low-low’ interactions, that is when the integral in (38) has cut-offs $\chi(\lambda|x - x_1|)$ and $\chi(\lambda|y_1 - y|)$, we note that in the two-dimensional Schrödinger equation, as in [31], one wishes to control integrals of the form

$$\int_0^\infty e^{it\lambda^2} \lambda \chi_1(\lambda) \mathcal{E}(\lambda) d\lambda$$

To this end, one integrates by parts once against the imaginary Gaussian to bound with

$$\frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} \left(\chi_1(\lambda) \mathcal{E}(\lambda) \right) \right| d\lambda$$

Much of the work is involved in finding a uniform bound for this resulting integral. In our case, we wish to bound integrals of the form

$$\int_0^\infty e^{it\lambda} (1 + \lambda) \chi_1(\lambda) \mathcal{E}(\lambda) d\lambda$$

Integrating by parts once, we are left to bound

$$\frac{1}{t} \int_0^\infty |\chi_1(\lambda) \mathcal{E}(\lambda)| + \left| \frac{d}{d\lambda} \left(\chi_1(\lambda) \mathcal{E}(\lambda) \right) \right| d\lambda \lesssim \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} \left(\chi_1(\lambda) \mathcal{E}(\lambda) \right) \right| d\lambda$$

The last inequality follows from (8) and (9). Defining

$$k(x, x_1) := 1 + \log^+ |x_1| + \log^- |x - x_1|,$$

with $\log^+ y = \chi_{\{y>1\}} \log y$ and $\log^- y = -\chi_{\{0<y<1\}} \log y$. The dispersive bound for Schrödinger, see Lemma 13 in [31], which proved the bound

$$(39) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda \chi_1(\lambda) Y_0(\lambda|x-x_1|) \chi(\lambda|x-x_1|) v(x_1) Q D_0 Q(x_1, y_1) \right. \\ \left. v(y_1) J_0(\lambda|y_1-y|) \chi(\lambda|x-x_1|) d\lambda \right| \lesssim \frac{k(x, x_1)}{t},$$

implies the bound for the wave equation. The desired bound of $t^{-\frac{1}{2}}$ follows from a simple interpolation with the boundedness in time of the integral. To show this, we use the orthogonality relation, $Qv = vQ = 0$ to control the logarithmic singularity of Y_0 . Specifically we note that we can replace $Y_0(\lambda|x-x_1|) \chi(\lambda|x-x_1|)$ in (38) with

$$(40) \quad F(\lambda, x, x_1) = Y_0(\lambda|x-x_1|) \chi(\lambda|x-x_1|) - \frac{2}{\pi} \chi(\lambda(1+|x|)) \log(\lambda(1+|x|)).$$

Noting Lemma 3.3 of [13] (which arose from the argument in [31]) we have the bounds on F

$$(41) \quad |F(\lambda, x, x_1)| \lesssim k(x, x_1), \quad |\partial_\lambda F(\lambda, x, x_1)| \lesssim \frac{1}{\lambda}.$$

The boundedness follows from the bounds on F and (8).

$$\int_0^\infty |\chi_1(\lambda) F(\lambda, x, x_1) J_0(\lambda|y-y_1|) \chi(\lambda|y-y_1|)| d\lambda \lesssim k(x, x_1) \int_0^1 d\lambda \lesssim k(x, x_1).$$

Thus we can conclude that

$$(42) \quad \left| \int_0^\infty e^{it\lambda} \chi_1(\lambda) Y_0(\lambda|x-x_1|) \chi(\lambda|x-x_1|) v(x_1) Q D_0 Q(x_1, y_1) \right. \\ \left. v(y_1) J_0(\lambda|y_1-y|) \chi(\lambda|x-x_1|) d\lambda \right| \lesssim \frac{k(x, x_1)}{t}$$

as desired.

Here we'll consider the integrals from the sine operator, as they are larger in the small λ regime. Consider the 'high-low' interaction terms,

$$\int_0^\infty e^{it\lambda} \chi_1(\lambda) J_0(\lambda d_1) \tilde{\chi}(\lambda d_1) Y_0(\lambda d_2) \chi(\lambda d_2) d\lambda = \int_0^\infty e^{it\lambda} \chi_1(\lambda) e^{-i\lambda d_1} \rho_+(\lambda d_1) F(\lambda, y, y_1) d\lambda.$$

Consider the case when $t - d_1 \geq t/2$, here we can safely integrate by parts to bound

$$\frac{1}{t-d_1} \int_0^\infty \left| \frac{d}{d\lambda} (\chi_1(\lambda) \rho_+(\lambda d_1) F(\lambda, y, y_1)) \right| d\lambda \lesssim \frac{k(y, y_1)}{t} \int_0^1 \frac{\lambda^{-\frac{3}{2}} \tilde{\chi}(\lambda d_1)}{d_1^{\frac{1}{2}}} d\lambda.$$

We note that the support condition implies that $\lambda d_1 \gtrsim 1$, so we can bound by

$$\frac{k(y, y_1) d_1^{\frac{1}{2}}}{t} \int_0^1 \lambda^{-\frac{1}{2}} d\lambda \lesssim \frac{k(y, y_1)}{t^{\frac{1}{2}}},$$

where we used that $d_1 \lesssim t$ in the last step.

If $t - d_1 \leq \frac{t}{2}$ then $d_1^{-1} \lesssim t^{-1}$ and we need to bound an integral of the form

$$\int_0^1 \frac{F(\lambda, y, y_1)}{(1 + \lambda d_1)^{\frac{1}{2}}} d\lambda \lesssim \int_0^1 \frac{k(y, y_1)}{d_1^{\frac{1}{2}} \lambda^{\frac{1}{2}}} d\lambda \lesssim \frac{k(y, y_1)}{t^{\frac{1}{2}}}.$$

The case where $J_0(\lambda d_1)$ is supported on small energy is bounded in the same way as $|J_0(\lambda d_1)\chi(\lambda d_1)| \lesssim 1 \lesssim k(x, x_1)$.

We now consider the ‘high-high’ terms. Here we do not take advantage of the cancellation between ‘+’ and ‘-’ resolvents, but instead bound the contribution of the $R_0^+ R_0^+$ and $R_0^- R_0^-$ terms individually. For the contribution of $R_0^- R_0^-$, using (10), we need to bound an integral of the form

$$\int_0^\infty e^{it\lambda} \chi_1(\lambda) e^{-i\lambda(d_1+d_2)} \rho_+(\lambda d_1) \rho_+(\lambda d_2) d\lambda$$

In the case that $t - (d_1 + d_2) \geq \frac{t}{2}$ we integrate by parts to bound integrals of the form

$$\frac{1}{t} \int_{\lambda \gtrsim d_1} \frac{d_1}{(1 + \lambda d_1)^{\frac{3}{2}} (1 + \lambda d_2)^{\frac{1}{2}}} d\lambda \lesssim \frac{1}{t} \frac{1}{d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}} \int_{\lambda \gtrsim d_1} \frac{1}{\lambda^2} d\lambda \lesssim \frac{1}{t} \frac{d_1^{\frac{1}{2}}}{d_2^{\frac{1}{2}}} \lesssim \frac{1}{t^{\frac{1}{2}}} \frac{1}{d_2^{\frac{1}{2}}}$$

Where we used that $t \gtrsim d_1, d_2$ in the last line. There is, of course, a similar term to consider with d_1 and d_2 switched.

On the other hand, if $t - (d_1 + d_2) \leq \frac{t}{2}$ we have that $t \lesssim \max(d_1, d_2)$ and we need to control an integral of the form

$$\int_0^1 \frac{1}{(1 + \lambda d_1)^{\frac{1}{2}} (1 + \lambda d_2)^{\frac{1}{2}}} d\lambda \lesssim \frac{1}{t^{\frac{1}{2}}} \int_0^1 \lambda^{-\frac{1}{2}} d\lambda \lesssim \frac{1}{t^{\frac{1}{2}}}.$$

This follows from the bound $t \lesssim \max(d_1, d_2)$ since

$$\frac{1}{(1 + \lambda d_1)^{\frac{1}{2}} (1 + \lambda d_2)^{\frac{1}{2}}} \lesssim \frac{1}{\lambda^{\frac{1}{2}} \max(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}})} \lesssim \frac{1}{t^{\frac{1}{2}} \lambda^{\frac{1}{2}}}.$$

We again note that the case of the positive phase, $R_0^+ R_0^+$ follows implicitly from these arguments with $2t$ replacing $\frac{t}{2}$ in the bounds for the two different cases analyzed when a ‘high’ term is involved in the interaction. We close the argument by noting that

$$\begin{aligned} \sup_{x, y \in \mathbb{R}^2} \int_{\mathbb{R}^4} k(x, x_1) v(x_1) |QD_0Q|(x_1, y_1) v(y_1) \frac{1}{|y_1 - y|^{\frac{1}{2}}} dx_1 dy_1 \\ \lesssim \sup_{x, y \in \mathbb{R}^2} \|k(x, \cdot) v(\cdot)\|_{L^2} \| |QD_0Q| \|_{L^2 \rightarrow L^2} \|v(\cdot) (1 + |\cdot - y|^{-\frac{1}{2}})\|_{L^2} \lesssim 1. \end{aligned}$$

□

We now turn to the contribution of the operator S in (37).

Lemma 4.9. *We have the bound*

$$(43) \quad \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda} \chi_1(\lambda) \left(\frac{R_0^+(\lambda^2)(x, x_1) v(x_1) S(x_1, y_1) v(y_1) R_0^+(\lambda^2)(y_1, y)}{h_+(\lambda)} - \frac{R_0^-(\lambda^2)(x, x_1) v(x_1) S(x_1, y_1) v(y_1) R_0^-(\lambda^2)(y_1, y)}{h_-(\lambda)} \right) d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}}$$

uniformly in x and y .

Proof. The ‘low-low’ interaction is bounded using Lemma 17 of [31] and the discussion at the start of Proposition 4.8. For the other terms we do not use the difference of ‘+’ and ‘-’ terms but estimate them individually. We note the bound

$$(44) \quad \chi_1(\lambda)\chi(\lambda|x-x_1|)\log(\lambda|x-x_1|) \lesssim (1+|\log \lambda|)(1+\log^-|x-x_1|).$$

This bound can be seen by considering the cases of $|x-x_1| < 1$ and $|x-x_1| > 1$ separately. Combining the above bound along with the fact that on the support of the integrals we have $|\partial_\lambda^k h_\pm^{-1}(\lambda)| \lesssim \lambda^{-k}|\log \lambda|^{-1}$ for $k = 0, 1$ which allows us to run through the same argument used for QD_0Q . As usual we consider the more delicate case when $R_0^-(\lambda^2)$ is involved. We first consider the ‘high-low’ interaction in which we wish to control

$$\int_0^\infty e^{it\lambda}\chi_1(\lambda)e^{-i\lambda d_1}\rho_+(\lambda d_1)\frac{\rho_-(\lambda d_2)}{h(\lambda)}d\lambda$$

In the first case, when $t-d_1 \geq \frac{t}{2}$. We first note that the integral above is bounded by $k(y, y_1)$, since

$$\begin{aligned} \int_0^\infty e^{it\lambda}\chi_1(\lambda)e^{-i\lambda d_1}\rho_+(\lambda d_1)\frac{\rho_-(\lambda d_2)}{h(\lambda)}d\lambda &\lesssim \int_0^{\lambda_1} \frac{(1+|\log \lambda|)(1+\log^-|x-x_1|)}{|h^-(\lambda)|}d\lambda \\ &\lesssim k(y, y_1) \int_0^{\lambda_1} |\log \lambda| + |\log \lambda|^{-1}d\lambda \lesssim k(y, y_1). \end{aligned}$$

We can interpolate between this and the bound obtained by integrating by parts against the combined phase $e^{i\lambda(t-d_1)}$ and use that $(t-d_1)^{-1} \lesssim t^{-1}$.

$$\frac{1}{t-d_1} \int_0^\infty \left| \frac{d}{d\lambda} \left(\frac{\rho_+(\lambda d_1)\rho_-(\lambda d_2)}{h(\lambda)} \right) \right| d\lambda \lesssim \frac{k(y, y_1)}{t} \int_{\lambda \gtrsim d_1^{-1}} d_1^{-\frac{1}{2}} \lambda^{-\frac{3}{2}} d\lambda \lesssim \frac{k(y, y_1)}{t}.$$

In the other case when $t-d_1 \leq \frac{t}{2}$, we have $t \lesssim d_1$, we use the decay of ρ_+ to bound by

$$\int_0^\infty \left| \chi_1(\lambda) \frac{(1+|\log \lambda|)k(y, y_1)}{\lambda^{\frac{1}{2}}d_1^{\frac{1}{2}}} \right| d\lambda \lesssim \frac{k(y, y_1)}{t^{\frac{1}{2}}} \int_0^{\lambda_1} \frac{1+|\log \lambda|}{\lambda^{\frac{1}{2}}} d\lambda \lesssim \frac{k(y, y_1)}{t^{\frac{1}{2}}}.$$

The ‘high-high’ interaction is handled similarly by considering the size of $t-(d_1+d_2)$ compared to $\frac{t}{2}$ as in the ‘high-high’ interactions considered in Proposition 4.8. As in the previous proofs, when $R_0^+(\lambda^2)$ is involved one needs only do the case analysis by comparing the size $t+d_1$ or $t+d_1+d_2$ with $2t$. □

Finally we turn to the error term, $E^\pm(\lambda)$ in (37). In this case, we note that all the functions of λ and their derivatives are smaller than those encountered in the QD_0Q and/or S terms.

Lemma 4.10. *We have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda}\chi_1(\lambda)R_0^\pm(\lambda^2)(x, x_1)v(x_1)E^\pm(\lambda)(x_1, y_1)v(y_1)R_0^\pm(\lambda^2)(y_1, y)d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. We provide a brief outline of the proof the convenience of the reader. We again reduce the analysis to a series of cases, ‘low-low’, ‘high-low’ and ‘high-high’ as in the proof of Proposition 4.8. Here, we use the vanishing of the error term at $\lambda = 0$ of the error term, see Lemma 4.6, allows us to control the logarithmic singularities of the resolvent. We cannot utilize the orthogonality relations we used for QD_0Q or the difference of ‘+’ and ‘-’ terms, though the smallness of $E^\pm(\lambda)$ more than compensates for this loss.

We provide a sketch of how one controls the ‘high-low’ interaction for the convenience of the reader. That is, we wish to control integrals of the form

$$\int_0^\infty e^{it\lambda} \chi_1(\lambda) e^{-i\lambda d_1} \rho_+(\lambda d_1) E^\pm(\lambda) \rho_-(\lambda d_2) d\lambda.$$

Using (44) it is easy to see the above integral is bounded by $(1 + \log^- d_2)$. That is we bound with

$$\sup_{0 < \lambda < \lambda_1} |\lambda^{-\frac{1}{2}} E^\pm(\lambda)| \int_0^{\lambda_1} |\lambda^{\frac{1}{2}} \rho_+(\lambda d_1) \rho_-(\lambda d_2)| d\lambda \lesssim (1 + \log^- d_2) \sup_{0 < \lambda < \lambda_1} |\lambda^{-\frac{1}{2}} E^\pm(\lambda)|.$$

By Lemma 4.6, the error term defines a bounded operator on L^2 .

As usual, for the time decay we have two cases to consider. If $t - d_1 \geq \frac{t}{2}$ we integrate by parts and note that

$$\begin{aligned} & \left| \frac{d}{d\lambda} \left(\chi_1(\lambda) \rho_+(\lambda d_1) \rho_-(\lambda d_2) \right) \right| \\ & \lesssim \frac{k(x, x_1)(1 + |\log \lambda|)}{\lambda^{\frac{1}{2}}} \left(\sup_{0 < \lambda < \lambda_1} |\lambda^{-\frac{1}{2}} E^\pm(\lambda)| + \sup_{0 < \lambda < \lambda_1} |\lambda^{\frac{1}{2}} \partial_\lambda E^\pm(\lambda)| \right). \end{aligned}$$

By Lemma 4.6 the operators involving the error term are absolutely bounded. The t^{-1} bound follows from the observations that $\lambda^{-\frac{1}{2}}(1 + |\log \lambda|)$ is integrable on the support of χ_1 and $k(x, x_1)v(x_1) \in L^2_{x_1}$ uniformly in x .

The case when $t - d_1 \leq \frac{t}{2}$ is simpler. Here we use $t \lesssim d_1$ to bound with

$$\frac{k(x, x_1)}{t^{\frac{1}{2}}} \int_0^\infty \chi_1(\lambda) |\lambda^{-\frac{1}{2}} E^\pm(\lambda)| (1 + |\log \lambda|) d\lambda$$

The bound follows from Lemma 4.6 as in the previously considered case. The ‘high-high’ interaction is handled similarly by considering the size of $t - (d_1 + d_2)$.

We note that the ‘low-low’ interaction is handled by Lemma 18 in [31] along with the discussion in the proof of Proposition 4.8. This t^{-1} bound is interpolated with the clear boundedness, due to the integrability of $\lambda^{\frac{1}{2}}(\log \lambda)^2$ on the support of χ_1 by treating the error term as in the mixed ‘high-low’ case above. □

Proof of Proposition 4.1. The proposition follows from the expansion (37), Lemma 4.7, Proposition 4.8, Lemmas 4.9, 4.10. □

5. WEIGHTED ESTIMATES

In this section, we prove a weighted version of the dispersive decay. At the cost of polynomially growing spatial weights, we can gain decay in t to achieve a decay rate of $t^{-1}(\log t)^{-2}$ for large t , which is integrable as $t \rightarrow \infty$. Such integrable dispersive bounds have applications in the analysis of non-linear equations, see [8, 29, 33, 37] for example. This result is obtained in a manner similar to the weighted decay for the two-dimensional Schrödinger equation obtained in [28, 13], and is motivated by recent work of Kopylova in [23].

Roughly speaking, we are taking advantage of the fact that the free Schrödinger operator $H_0 = -\Delta$ has a resonance at zero energy in dimension two. It is well-known that a resonance at zero energy slows the long-time decay rate of the evolution operator, see [19, 15, 24, 13] and Section 6 below. By perturbing with a potential that removes this zero energy resonance, one can obtain a faster time decay rate by allowing for spatially growing weights. A similar gain of time decay rate can be seen in one spatial dimension, [32].

The condition of zero being regular is equivalent to the boundedness of the operators $R_V^\pm(\lambda^2)$ between certain weighted L^2 spaces. We can see from the expansion in Lemma 4.2 that zero is not regular for the free resolvent due to the singular $\log \lambda$ term.

Due to the representation (5), Theorem 1.2 follows from the following oscillatory integral bound.

Proposition 5.1. *If $|V(x)| \lesssim \langle x \rangle^{-3-\alpha}$ for some $\alpha \in (0, \frac{1}{4})$ and zero is a regular point of the spectrum of $H = -\Delta + V$, then for $t > 2$,*

$$\left| \int_0^\infty (\sin(t\lambda)\langle \lambda \rangle^{-\frac{1}{4}-} + \lambda \cos(t\lambda)\langle \lambda \rangle^{-\frac{3}{4}-}) [R_V^+(\lambda^2)(x, y) - R_V^-(\lambda^2)(x, y)] d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t(\log t)^2}.$$

The proof of this bound follows in a similar manner as the non-weighted $t^{-\frac{1}{2}}$ bound. Accordingly, we divide this section into two subsections. We first control the high energy, when $\lambda > \lambda_1 > 0$ with a faster time decay when allowing for weights, then we state revised resolvent expansions from [14] for low energy that allow us to prove the desired time decay for low energy.

5.1. High energy. Much of the care we took in Section 3 was to avoid growth in the spatial variables by avoiding differentiating the phase $e^{i\lambda|x-y|}$. If we allow for a spatially weighted estimate, one can proceed in a less delicate manner.

As usual, the high energy portion of Proposition 5.1 follows from

Proposition 5.2. *If $|V(x)| \lesssim \langle x \rangle^{-3-\alpha}$ for some $\alpha \in (0, \frac{1}{4})$ and zero is a regular point of the spectrum of $H = -\Delta + V$, then for $t > 2$,*

$$\left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} [R_V^+(\lambda^2)(x, y) - R_V^-(\lambda^2)(x, y)] d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}.$$

The high energy bounds follow with some modifications to Lemma 3.3 and 3.4 which are outlined below. We again start with the resolvent expansion (15). We consider the contribution of the free

resolvent with the difference of the ‘+’ and ‘-’ terms so that for the first term in the Born series, (15), we need only consider the contribution of the Bessel function J_0 .

Lemma 5.3. *For $t > 2$, we have the bound*

$$\sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} J_0(\lambda|x-y|) d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}}}{t^{\frac{3}{2}}} + \frac{1}{t^2}.$$

Proof. We first consider the J_0 on $\lambda|x-y| \lesssim 1$. By the expansion (8), we need only bound

$$\int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) d\lambda$$

We note that χ' is supported on $\lambda \approx 1$ and we have

$$|\partial_\lambda^k \tilde{\chi}_1(\lambda)|, |\partial_\lambda^k \chi_L(\lambda)| \lesssim \lambda^{-k}.$$

Integrating by parts twice against the phase $e^{it\lambda}$ yields the bound of

$$\frac{1}{t^2} \int_0^\infty \left| \frac{d}{d\lambda} \left(\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \tilde{O}(1) \right) \right| d\lambda \lesssim \frac{1}{t^2} \int_1^\infty \lambda^{-\frac{5}{2}-} d\lambda \lesssim t^{-2}.$$

We next consider when $\lambda|x-y| \gtrsim 1$ by (10) we need to bound

$$\int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} e^{\pm i\lambda|x-y|} \omega_\pm(\lambda|x-y|) d\lambda$$

Recall that $\omega_\pm(z) = \tilde{O}(z^{-\frac{1}{2}})$ and is supported on $z \gtrsim 1$. We consider the case of the ‘+’ phase along with the case of the ‘-’ phase when $t - |x-y| \geq \frac{t}{2}$ together. In both cases, we can integrate by parts twice against the phase $e^{i\lambda(t \pm |x-y|)}$ safely and bound with

$$\frac{1}{t^2} \int_0^\infty \left| \frac{d^2}{d\lambda^2} \left(\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} \omega_\pm(\lambda|x-y|) \right) \right| d\lambda \lesssim t^{-2} \int_1^\infty \lambda^{-\frac{5}{2}-} d\lambda \lesssim t^{-2}.$$

For the ‘-’ phase, we consider the second case in which $t - |x-y| \leq \frac{t}{2}$, in which case $t \lesssim |x-y|$.

Here we integrate by parts once against $e^{it\lambda}$ and bound with

$$\begin{aligned} \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} \left(\tilde{\chi}_1(\lambda) \chi_L(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda|x-y|} \omega_-(\lambda|x-y|) \right) \right| d\lambda &\lesssim \frac{1+|x-y|}{t|x-y|^{\frac{1}{2}}} \int_1^\infty \lambda^{-1-} d\lambda \\ &\lesssim \frac{1+|x-y|^{\frac{1}{2}}}{t^{\frac{3}{2}}} \lesssim \frac{\langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}}}{t^{\frac{3}{2}}}. \end{aligned}$$

Here we see why polynomial weights are needed to gain the extra time decay. □

For the remaining terms in (15) we note that Lemmas 3.3 and 3.4 ensure the convergence of the λ integrals we wish to analyze. Accordingly, we can omit the χ_L cut-off function. We now turn to the second term of the Born series (15) and estimate the ‘+’ and ‘-’ terms separately.

Lemma 5.4. *If $|V(x)| \lesssim \langle x \rangle^{-3-\alpha}$, then for $t > 2$ we have the bound*

$$\left| \int_{\mathbb{R}^2} \int_0^\infty e^{it\lambda} \lambda^{-\frac{1}{2}-} \tilde{\chi}_1(\lambda) R_0^\pm(\lambda^2)(x, z) V(z) R_0^\pm(\lambda^2)(z, y) d\lambda dz \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha} \langle y \rangle^{\frac{1}{2}+\alpha}}{t^{1+\alpha}}.$$

Proof. This proof follows along the lines of the proof of Lemma 3.3. We recall the expansion for $R_0^\pm(\lambda^2)(x, y)$ given in (17). For the ‘low-low’ interaction, integrating by parts twice against the phase $e^{it\lambda}$ yields

$$\begin{aligned} \left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2) d\lambda \right| &\lesssim \frac{1}{t^2} \int_0^\infty \left| \frac{d^2}{d\lambda^2} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_1) \rho_-(\lambda d_2)) \right| d\lambda \\ &\lesssim \frac{(1 + \log^- d_1)(1 + \log^- d_2)}{t^2} \int_1^\infty \lambda^{-\frac{5}{2}-} d\lambda \lesssim \frac{(1 + \log^- d_1)(1 + \log^- d_2)}{t^2}. \end{aligned}$$

Here we used the inequality (19) to bound with the $\log^- d_j$ spatial weights. For the ‘high-low’ and ‘high-high’ interactions, we consider only the negative phase contributions. For the ‘high-low’ interaction, we again consider cases. If $t - d_1 \geq \frac{t}{2}$, then we can integrate by parts against the combined phase $e^{i\lambda(t-d_1)}$,

$$\begin{aligned} \left| \int_0^\infty e^{i\lambda(t-d_1)} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_2) \rho_+(\lambda d_1) d\lambda \right| \\ \lesssim \frac{1}{(t-d_1)^2} \int_0^\infty \left| \frac{d^2}{d\lambda^2} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_2) \rho_+(\lambda d_1)) \right| d\lambda \\ \lesssim \frac{1 + \log^- d_2}{t^2 d_1^{\frac{1}{2}}} \int_1^\infty \lambda^{-\frac{5}{2}-} d\lambda \lesssim \frac{1 + \log^- d_2}{t^2 d_1^{\frac{1}{2}}}. \end{aligned}$$

On the other hand, if $t - d_1 \leq \frac{t}{2}$ then we integrate by parts once against the phase $e^{it\lambda}$ and, similar to the case of the free resolvent only, use the decay of the Bessel function and that $t \lesssim d_1$ to see

$$\begin{aligned} \left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_2) e^{-i\lambda d_1} \rho_+(\lambda d_1) d\lambda \right| \\ \lesssim \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_-(\lambda d_2) e^{-i\lambda d_1} \rho_+(\lambda d_1)) \right| d\lambda \\ \lesssim \frac{(1 + \log^- d_2) \langle d_1 \rangle}{t d_1^{\frac{1}{2}}} \int_1^\infty \lambda^{-1-} d\lambda \lesssim \frac{(1 + \log^- d_2) \langle x \rangle^{\frac{1}{2}} \langle z \rangle^{\frac{1}{2}}}{t^{\frac{3}{2}}}. \end{aligned}$$

There is a corresponding term with d_1 and d_2 interchanged. The ‘high-high’ interaction is handled in cases as well. If $t - (d_1 + d_2) \geq \frac{t}{2}$ we use that

$$\begin{aligned} \left| \int_0^\infty e^{i\lambda(t-(d_1+d_2))} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_+(\lambda d_2) \rho_+(\lambda d_1) d\lambda \right| \\ \lesssim \frac{1}{t^2} \int_0^\infty \left| \frac{d^2}{d\lambda^2} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} \rho_+(\lambda d_2) \rho_+(\lambda d_1)) \right| d\lambda \lesssim \frac{1}{t^2 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}} \int_1^\infty \lambda^{-\frac{5}{2}-} d\lambda \lesssim \frac{1}{t^2 d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}}. \end{aligned}$$

On the other hand, if $t - (d_1 + d_2) \leq \frac{t}{2}$, then $t \lesssim \max(d_1, d_2)$ and we see that

$$\begin{aligned} \left| \int_0^\infty e^{it\lambda} \tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda(d_1+d_2)} \rho_+(\lambda d_2) \rho_+(\lambda d_1) d\lambda \right| \\ \lesssim \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} (\tilde{\chi}_1(\lambda) \lambda^{-\frac{1}{2}-} e^{-i\lambda(d_1+d_2)} \rho_+(\lambda d_2) \rho_+(\lambda d_1)) \right| d\lambda \end{aligned}$$

$$\lesssim \frac{1 + d_1 + d_2}{td_1^{\frac{1}{2}}d_2^{\frac{1}{2}}} \int_1^\infty \lambda^{-1-} d\lambda \lesssim \frac{1 + \max(d_1, d_2)}{t \min(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}}) \max(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}})} \lesssim \frac{(\langle x \rangle \langle z \rangle \langle y \rangle)^{\frac{1}{2} + \alpha}}{t^{1+\alpha} \min(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}})}.$$

To conclude the desired bounds, we note that under the assumptions of Proposition 5.1

$$\int_{\mathbb{R}^2} |V(z)| \left((1 + \log^- |x - z|)(1 + |z - y|^{-\frac{1}{2}}) + \langle z \rangle^{\frac{1}{2} + \alpha} |x - z|^{-\frac{1}{2}} + (|x - z|^{-\frac{1}{2}} |z - y|^{-\frac{1}{2}}) \right) dz \lesssim 1$$

uniformly in x and y . □

Proof of Proposition 5.2. The Proposition follows from Lemma 5.3, Lemma 5.4 and the following modification of Lemma 3.4 in Section 3. For the tail of the Born series, we note that we can essentially repeat the argument of Lemma 3.4 but integrate by parts twice in (31) to see

$$|(31)| \lesssim \frac{1}{t^2} \int_0^\infty \left| \frac{d^2}{d\lambda^2} \left(\lambda^{-\frac{1}{2}-} \mathcal{E}(\lambda)(x, y) \right) d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}}}{t^2}.$$

Here we use the bound (27) directly instead of separating the phase from the large λr portion. This, of course, leaves us with polynomial weights and requires an extra power of decay on the potential. This is easily seen since differentiating the resolvents twice requires a potential that maps $L^{2, -\frac{5}{2}-}$ to $L^{2, \frac{1}{2}+}$. □

5.2. Low Energy. To attain the extra time decay, we need to modify slightly the resolvent expansions laid out in Section 4. Accordingly, we use the resolvent expansions developed for the analysis of the Schrödinger equation from [13].

Proposition 5.5. *Fix $0 < \alpha < \frac{1}{4}$. Let $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$. For any $t > 2$, we have*

$$(45) \quad \left| \int_0^\infty (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \\ \lesssim \frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{1}{2} + \alpha} \langle y \rangle^{\frac{1}{2} + \alpha}}{t^{1+\alpha}}.$$

The following corollary of Lemma 4.2 follows from the bounds for $\partial_\lambda E_0^\pm$ and $\partial_\lambda^2 E_0^\pm$.

Corollary 5.6. *For $0 < \alpha < 1$ and $b > a > 0$ we have*

$$|\partial_\lambda E_0^\pm(b) - \partial_\lambda E_0^\pm(a)| \lesssim a^{-\frac{1}{2}} |b - a|^\alpha |x - y|^{\frac{1}{2} + \alpha}.$$

Proof. The bounds on $\partial_\lambda^2 E_0^\pm$ from Lemma 4.2 along with the mean value theorem guarantee that

$$|\partial_\lambda E_0^\pm(b) - \partial_\lambda E_0^\pm(a)| \lesssim a^{-\frac{1}{2}} |b - a| |x - y|^{\frac{3}{2}}.$$

The statement of this corollary then follows from a simple interpolation with the bounds in Lemma 4.2. □

This corollary implies the improved error bounds below.

Lemma 5.7. *Let $0 < \alpha < 1$. The error term in Lemma 4.3 satisfies the bound*

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E_1^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \\ & \quad + \left\| \sup_{0 < \lambda < b < \lambda_1} \lambda^{\frac{1}{2}} (b - \lambda)^{-\alpha} |\partial_\lambda E_1^\pm(b) - \partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$.

Recall that from Lemma 4.6 we have the expansion

$$M^\pm(\lambda)^{-1} = h_\pm(\lambda)^{-1} S + QD_0Q + E^\pm(\lambda),$$

Lemma 5.8. *Let $0 < \alpha < 1$. Under the conditions of Lemma 4.6, we have the improved error bounds*

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \right\|_{HS} \\ & \quad + \left\| \sup_{0 < \lambda < b \lesssim \lambda < \lambda_1} \lambda^{\frac{1}{2} + \alpha} (b - \lambda)^{-\alpha} |\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(a)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$.

We note that for fixed x, y the kernel $R_V^\pm(\lambda^2)(x, y)$ of the resolvent remains bounded as $\lambda \rightarrow 0$. This is because of a cancellation between the first and second summands of the second line of the expansion for R_V^\pm in (37). A consequence of this cancellation is cancellation of the slowest decaying terms in the evolution which makes it possible to obtain the faster time decay.

We first establish the following simple Lemma which allows us to use the Lipschitz bounds on the error terms from Lemma 5.8. This Lemma is crucial on two fronts, it allows us to minimize both the decay assumptions on the potential and the size of polynomial weights needed in Theorem 1.2, see Proposition 5.10, Lemma 5.13, Proposition 5.14, and Proposition 5.15 below.

Lemma 5.9. *Assume that $\mathcal{E}(0) = 0$. For $t > 2$, we have*

$$(46) \quad \left| \int_0^\infty e^{\pm it\lambda} \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda t)} d\lambda + \frac{1}{t} \int_{t-1}^\infty |\mathcal{E}'(\lambda + \pi/t) - \mathcal{E}'(\lambda)| d\lambda.$$

Proof. We integrate by parts once and then consider two different regions. We give the proof for the positive phase, the negative phase follows identically.

$$\int_0^\infty e^{it\lambda} \mathcal{E}(\lambda) d\lambda = -\frac{i}{t} \int_0^\infty e^{it\lambda} \mathcal{E}'(\lambda) d\lambda$$

We divide this integral into two pieces. The first region $0 < \lambda < \frac{2\pi}{t}$, can clearly be bounded by the first integral in the statement of the Lemma. On the second region, $\frac{2\pi}{t} < \lambda$, we note that

$$\int_{\frac{2\pi}{t}}^\infty e^{it\lambda} \mathcal{E}'(\lambda) d\lambda = - \int_{\frac{2\pi}{t}}^\infty e^{it(\lambda - \frac{\pi}{t})} \mathcal{E}'(\lambda) d\lambda = - \int_{\frac{\pi}{t}}^\infty e^{it\lambda} \mathcal{E}'(\lambda + \pi/t) d\lambda.$$

Therefore it suffices to consider (the integral on $[\pi/t, 2\pi/t]$ is bounded by the first integral on the right hand side of (46))

$$\int_{\frac{\pi}{t}}^{\infty} e^{it\lambda} \left(\mathcal{E}'(\lambda) - \mathcal{E}'(\lambda + \pi/t) \right) d\lambda.$$

□

We start with the contribution of the free resolvent to (45). In this instance we work with the functions $\sin(t\lambda)$ and $\lambda \cos(t\lambda)$ directly rather than estimating terms containing $e^{\pm i\lambda t}$. This will allow us to explicitly determine the slowest decaying (order t^{-1}) term.

Proposition 5.10. *For any $0 < \alpha < \frac{1}{4}$, we have*

$$\int_0^{\infty} (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) [R_0^+(\lambda^2) - R_0^-(\lambda^2)](x, y) d\lambda = \frac{i}{2t} + O\left(\frac{\langle x \rangle^{\frac{1}{2} + \alpha} \langle y \rangle^{\frac{1}{2} + \alpha}}{t^{1 + \alpha}}\right).$$

Proof. Using Lemma 4.2, we have

$$R_0^+ - R_0^- = \frac{i}{2} + E_0^+(\lambda) - E_0^-(\lambda).$$

We first consider the $\sin(t\lambda)$ term, the $\lambda \cos(t\lambda)$ term is smaller. We now rewrite the λ integral above as

$$\frac{i}{2} \int_0^{\infty} \sin(t\lambda) \chi_1(\lambda) d\lambda + \int_0^{\infty} \sin(t\lambda) \chi_1(\lambda) (E_0^+(\lambda) - E_0^-(\lambda)) d\lambda =: I + II.$$

Note that by integrating by parts we obtain

$$(47) \quad I = \frac{i}{2} \left(\frac{-\cos(t\lambda) \chi_1(\lambda)}{t} \Big|_0^{\infty} + \frac{1}{t} \int_0^{\infty} \cos(t\lambda) \chi_1'(\lambda) d\lambda \right) = \frac{i}{2t} + O(t^{-2})$$

Since χ_1' is supported on $\lambda \approx 1$, the second term can be integrated by parts again.

Using the bounds in Lemma 4.2 for $\mathcal{E}(\lambda) = \chi_1(\lambda)(E_0^+(\lambda) - E_0^-(\lambda))$, we see that $\mathcal{E}(0) = 0$, and from Corollary 5.6,

$$\begin{aligned} |\partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}} |x - y|^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \sqrt{\langle x \rangle \langle y \rangle}, \\ \left| \partial_\lambda \mathcal{E}(\lambda + \pi/t) - \partial_\lambda \mathcal{E}(\lambda) \right| &\lesssim \lambda^{-\frac{1}{2} - \alpha} t^{-\alpha} \langle x - y \rangle^{\frac{1}{2} + \alpha} \end{aligned}$$

Accordingly, we decompose sine and cosine into terms involving $e^{\pm i\lambda t}$ use Lemma 5.9 to obtain

$$|II| \lesssim \frac{1}{t} \int_0^{\infty} \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda t)} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^{\infty} |\mathcal{E}'(\lambda + \pi/t) - \mathcal{E}'(\lambda)| d\lambda.$$

Using the bounds above, we estimate the first integral by

$$\frac{\sqrt{\langle x \rangle \langle y \rangle}}{t} \int_0^{\infty} \frac{1}{\sqrt{\lambda}(1 + \lambda t)} d\lambda \lesssim \frac{\sqrt{\langle x \rangle \langle y \rangle}}{t^{1 + \alpha}}.$$

To estimate the second integral, we apply the Lipschitz bound to see

$$\frac{\langle x \rangle \langle y \rangle^{\frac{1}{2} + \alpha}}{t} \int_{t^{-1}}^{\lambda_1} \lambda^{-\frac{1}{2} - \alpha} (t\lambda)^{-\alpha} d\lambda \lesssim \frac{\langle x \rangle \langle y \rangle^{\frac{1}{2} + \alpha}}{t^{1 + \alpha}},$$

since $\alpha \in (0, \frac{1}{4})$.

For the cosine term we note that we need to control and integral of the form

$$\int_0^\infty \cos(t\lambda) \mathcal{E}_1(\lambda) d\lambda,$$

with $\mathcal{E}_1(\lambda) = \lambda \chi_1(\lambda) (\frac{i}{2} + E_0^+(\lambda) - E_0^-(\lambda))$. Here, we can integrate by parts without an order t^{-1} boundary term. This follows from the vanishing of $\mathcal{E}_1(\lambda)$ and at $\lambda = 0$. With the bounds on E_0^\pm we can bound this contribution as we did term II. \square

We now consider the contribution of the second term in (37) to (45):

$$(48) \quad \int_{\mathbb{R}^4} \int_0^\infty (\sin(t\lambda) + \cos(t\lambda)\lambda) \chi_1(\lambda) [\mathcal{R}^- - \mathcal{R}^+] v(x_1) S(x_1, y_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$(49) \quad \mathcal{R}^\pm = \frac{R_0^\pm(\lambda^2)(x, x_1) R_0^\pm(\lambda^2)(y_1, y)}{h_\pm(\lambda)}.$$

Proposition 5.11. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, then we have*

$$(48) = -\frac{i}{2t} + O\left(\frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t(\log(t))^2}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha^+} \langle y \rangle^{\frac{1}{2}+\alpha^+}}{t^{1+\alpha}}\right).$$

We will prove this Proposition through a series of Lemmas to control the various terms that arise from the difference of the ‘+’ and ‘-’ terms. Recall from Lemma 4.2 that

$$R_0^\pm(\lambda^2)(x, x_1) = \frac{1}{\|V\|_1} g^\pm(\lambda) + G_0(x, x_1) + E_0^\pm(\lambda)(x, x_1).$$

Recalling that $h^\pm(\lambda) = g^\pm(\lambda) + c$ with $c \in \mathbb{R}$, we see

$$\mathcal{R}^\pm = \frac{1}{\|V\|_1^2} [g^\pm(\lambda) + c + \tilde{G}_0(x, x_1) + \tilde{G}_0(y, y_1) + \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{g^\pm(\lambda) + c}] + E_2^\pm(\lambda),$$

where

$$(50) \quad E_2^\pm(\lambda) := \frac{1}{\|V\|_1} \left(1 + \frac{\tilde{G}_0(x, x_1)}{g^\pm(\lambda) + c}\right) E_0^\pm(\lambda)(y, y_1) + \frac{1}{\|V\|_1} \left(1 + \frac{\tilde{G}_0(y, y_1)}{g^\pm(\lambda) + c}\right) E_0^\pm(\lambda)(x, x_1) \\ + \frac{E_0^\pm(\lambda)(x, x_1) E_0^\pm(\lambda)(y, y_1)}{g^\pm(\lambda) + c},$$

and $\tilde{G}_0 = \|V\|_1 G_0 - c$. Using this and (34), we have

$$\mathcal{R}^- - \mathcal{R}^+ = -\frac{i}{2\|V\|_1} + c_3 \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{(\log(\lambda) + c_1)^2 + c_2^2} + E_2^-(\lambda) - E_2^+(\lambda),$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

We consider first the contribution of the sine, the contribution of the cosine term can be controlled similarly but without the boundary term due to the additional power of λ , see Proposition 5.10. We rewrite the λ integral in (48) as a sum of

$$(51) \quad -\frac{i}{2\|V\|_1} \int_0^\infty \sin(t\lambda) \chi_1(\lambda) d\lambda,$$

$$(52) \quad \int_0^\infty \sin(t\lambda) \chi_1(\lambda) \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{(\log(\lambda) + c_1)^2 + c_2^2} d\lambda,$$

$$(53) \quad \int_0^\infty \sin(t\lambda) \chi_1(\lambda) [E_2^-(\lambda) - E_2^+(\lambda)] d\lambda.$$

Note that by (47) we have

$$(54) \quad (51) = -\frac{i}{2t\|V\|_1} + O(t^{-2}).$$

The leading term above will cancel the boundary term that arose in Proposition 5.10.

The decay rate $\frac{1}{t \log^2(t)}$ appears in the weighted H^s setting in [23] and appears in our analysis due to the contribution of (52), as shown in the following lemma.

Lemma 5.12. *For $t > 2$, we have the bound*

$$|(52)| \lesssim \frac{1}{t \log^2(t)} k(x, x_1) k(y, y_1) (1 + \log^+ |x|) (1 + \log^+ |y|).$$

Proof. First note that

$$(55) \quad |\tilde{G}_0(x, x_1)| \lesssim 1 + |\log |x - x_1|| \lesssim k(x, x_1) (1 + \log^+ |x|).$$

Second, we bound the λ -integral by integrating by parts. Denote $\mathcal{E}(\lambda) = \frac{\chi_1(\lambda)}{(\log(\lambda) + c_1)^2 + c_2^2}$. We note that $\mathcal{E}(0) = 0$ and satisfies the following bounds

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \frac{\chi_1(\lambda)}{\lambda |\log(\lambda)|^3}, \quad \left| \partial_\lambda^2 \mathcal{E}(\lambda) \right| \lesssim \frac{\chi_1(\lambda)}{\lambda^2 |\log(\lambda)|^3}.$$

Using these bounds, we will integrate by parts once, then divide the integral into two pieces. On the first region we consider $0 < \lambda < t^{-1}$ and perform another integration by parts on the second region, $\lambda > t^{-1}$.

$$\begin{aligned} \left| \int_0^\infty \sin(t\lambda) \mathcal{E}(\lambda) d\lambda \right| &\lesssim \frac{1}{t} \int_0^{t^{-1}} |\mathcal{E}'(\lambda)| d\lambda + \left| \frac{\mathcal{E}'(t^{-1})}{t^2} \right| + \frac{1}{t^2} \int_{t^{-1}}^\infty |\mathcal{E}''(\lambda)| d\lambda \\ &\lesssim \frac{1}{t} \int_0^{t^{-1}} \frac{1}{\lambda |\log(\lambda)|^3} d\lambda + \frac{1}{t |\log(t)|^3} + \frac{1}{t^2} \int_{t^{-1}}^\infty \frac{1}{\lambda^2 |\log(\lambda)|^3} d\lambda. \end{aligned}$$

A simple calculation shows that

$$\frac{1}{t} \int_0^{t^{-1}} \frac{1}{\lambda |\log(\lambda)|^3} d\lambda \sim \frac{1}{t |\log(t)|^2}.$$

Finally we bound the integral on $[t^{-1}, \infty)$:

$$\begin{aligned} \frac{1}{t^2} \int_{t^{-1}}^\infty \frac{\chi_1(\lambda)}{\lambda^2 |\log(\lambda)|^3} d\lambda &\lesssim \frac{1}{t^2} + \frac{1}{t^2} \int_{t^{-1}}^{1/2} \frac{1}{\lambda^2 |\log(\lambda)|^3} d\lambda \\ &\lesssim \frac{1}{t^2} + \frac{1}{t^2} \int_{t^{-1/2}}^{1/2} \frac{1}{\lambda^2} d\lambda + \frac{1}{t^2} \int_{t^{-1}}^{t^{-1/2}} \frac{1}{\lambda^2 |\log(t)|^3} d\lambda \lesssim \frac{1}{t^2} + \frac{1}{t |\log(t)|^3}. \end{aligned}$$

The first inequality follows since the integral converges on $[\frac{1}{2}, \infty)$.

Combining the bounds we obtained above finishes the proof of the lemma. \square

Lemma 5.13. *Let $0 < \alpha < 1/4$. For $t > 2$, we have the bound*

$$|(53)| \lesssim t^{-1-\alpha} k(x, x_1) k(y, y_1) (\langle x \rangle \langle y \rangle \langle x_1 \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha+}.$$

Proof. We will only consider the following part of (53):

$$(56) \quad \int_0^\infty e^{it\lambda} \chi_1(\lambda) \left(1 + \frac{\tilde{G}_0(x, x_1)}{g(\lambda) + c}\right) E_0(\lambda)(y, y_1) d\lambda =: \int_0^\infty e^{it\lambda} \mathcal{E}(\lambda) d\lambda.$$

The other parts (and their derivatives) are either of this form or much smaller. We also omit the \pm signs since we can not rely on a cancellation between ‘+’ and ‘-’ terms.

Using Lemma 4.2, Corollary 5.6, and (55), we estimate (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim k(x, x_1) (1 + \log^+ |x|) \chi_1(\lambda) \lambda^{-\frac{1}{2}} \langle y - y_1 \rangle^{\frac{1}{2}} \lesssim k(x, x_1) (1 + \log^+ |x|) \sqrt{\langle y \rangle \langle y_1 \rangle} \lambda^{-\frac{1}{2}},$$

and

$$\begin{aligned} |\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| &\lesssim \chi_1(\lambda) k(x, x_1) (1 + \log^+ |x|) \lambda^{-\frac{1}{2}-\alpha} (b - \lambda)^\alpha \langle y - y_1 \rangle^{\frac{1}{2}+\alpha} \\ &\lesssim \chi_1(\lambda) k(x, x_1) (1 + \log^+ |x|) (\langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha} \lambda^{-\frac{1}{2}-\alpha} (b - \lambda)^\alpha. \end{aligned}$$

Noting that $\mathcal{E}(0) = 0$ we can use Lemma 5.9 as in Proposition 5.10 to obtain

$$|(56)| \lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda t)} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda + \pi/t) - \mathcal{E}'(\lambda)| d\lambda.$$

Using the bounds above, we estimate the first integral by

$$\frac{k(x, x_1) (1 + \log^+ |x|) \sqrt{\langle y \rangle \langle y_1 \rangle}}{t} \int_0^\infty \frac{1}{\sqrt{\lambda} (1 + \lambda t)} d\lambda \lesssim \frac{k(x, x_1) (1 + \log^+ |x|) \sqrt{\langle y \rangle \langle y_1 \rangle}}{t^{1+\alpha}}.$$

To estimate the second integral, we apply the Lipschitz bound with $b = \lambda + \pi/t$ to see

$$\frac{k(x, x_1) (1 + \log^+ |x|) (\langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha}}{t} \int_{t^{-1}}^{\lambda_1} \lambda^{-\frac{1}{2}-\alpha} (t\lambda)^{-\alpha} d\lambda \lesssim \frac{k(x, x_1) (1 + \log^+ |x|) (\langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha}}{t^{1+\alpha}},$$

since $\alpha \in (0, \frac{1}{4})$.

Taking into account the contribution of the term with the roles of x and y switched, and the bound $1 + \log^+ |x| \lesssim \langle x \rangle^{0+}$, we obtain the assertion of the lemma. \square

We are now prepared to prove the Proposition.

Proof of Proposition 5.11. Using the bounds we obtained in (54), Lemma 5.12, Lemma 5.13 in (48), we obtain

$$\begin{aligned} (48) &= -\frac{i}{2t\|V\|_1} \int_{\mathbb{R}^4} v(x_1) S(x_1, y_1) v(y_1) dx_1 dy_1 \\ &\quad + O\left(\frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2(t)} \int_{\mathbb{R}^4} k(x, x_1) v(x_1) |S(x_1, y_1)| v(y_1) k(y, y_1) dx_1 dy_1\right) \\ &\quad + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}} \int_{\mathbb{R}^4} k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha+} v(x_1) |S(x_1, y_1)| v(y_1) k(y, y_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha+} dx_1 dy_1\right). \end{aligned}$$

Note that the integrals in the error terms are bounded in x, y , since

$$\sup_{y \in \mathbb{R}^2} \|v(y_1) \langle y_1 \rangle^{\frac{1}{2} + \alpha^+} k(y, y_1)\|_{L^2_{y_1}} \lesssim 1.$$

Also note that we can replace S with P in the first integral since the other parts of the operator S contains Q on at least one side and $Qv = vQ = 0$. Therefore,

$$\begin{aligned} (48) &= -\frac{i}{2t\|V\|_1} \int_{\mathbb{R}^4} v(x_1)P(x_1, y_1)v(y_1)dx_1dy_1 \\ &\quad + O\left(\frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2(t)}\right) + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{1}{2} + \alpha^+}}{t^{1+\alpha}}\right) \\ &= -\frac{i}{2t} + O\left(\frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2(t)}\right) + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{1}{2} + \alpha^+}}{t^{1+\alpha}}\right). \end{aligned}$$

□

Next we consider the contribution of the third term in (37) to (45):

$$(57) \quad \int_{\mathbb{R}^4} \int_0^\infty \sin(t\lambda) \chi_1(\lambda) [\mathcal{R}_2^- - \mathcal{R}_2^+] v(x_1) [QD_0Q](x_1, y_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$(58) \quad \mathcal{R}_2^\pm = R_0^\pm(\lambda^2)(x, x_1) R_0^\pm(\lambda^2)(y_1, y).$$

We notice that many of the terms that arise in this term have zero contribution due to either the difference of the ‘+’ and ‘-’ terms and the orthogonality of Q and v . Recall from Lemma 4.2 that

$$R_0^\pm(\lambda^2)(x, x_1) = c[a \log(\lambda|x - x_1|) + b \pm i] + E_0^\pm(\lambda)(x, x_1),$$

where $a, b, c \in \mathbb{R}$. Therefore

$$\begin{aligned} \mathcal{R}_2^\pm &= c^2[(a \log(\lambda|x - x_1|) + b)(a \log(\lambda|y - y_1|) + b) - 1] \\ &\quad \pm ic^2[a \log(\lambda|x - x_1|) + a \log(\lambda|y - y_1|) + 2b] + E_3^\pm(\lambda), \end{aligned}$$

where

$$(59) \quad \begin{aligned} E_3^\pm(\lambda) &:= c[a \log(\lambda|x - x_1|) + b \pm i] E_0^\pm(\lambda)(y, y_1) \\ &\quad + c[a \log(\lambda|y - y_1|) + b \pm i] E_0^\pm(\lambda)(x, x_1) + E_0^\pm(\lambda)(x, x_1) E_0^\pm(\lambda)(y, y_1). \end{aligned}$$

Using this, we have

$$\mathcal{R}_2^- - \mathcal{R}_2^+ = -2c^2(a \log(\lambda|x - x_1|) + a \log(\lambda|y - y_1|) + 2b) + E_3^-(\lambda) - E_3^+(\lambda).$$

Using this in (57), and noting that the contribution of the first summand vanishes since $Qv = vQ = 0$, we obtain

$$(60) \quad (57) = \int_{\mathbb{R}^4} \int_0^\infty \sin(t\lambda) \chi_1(\lambda) [E_3^-(\lambda) - E_3^+(\lambda)] v(x_1) [QD_0Q](x_1, y_1) v(y_1) d\lambda dx_1 dy_1.$$

Proposition 5.14. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, then we have*

$$|(57)| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}.$$

Proof. Let $\mathcal{E}(\lambda) = \chi_1(\lambda)E_3(\lambda)$ (we dropped the ‘ \pm ’ signs). Using

$$|\log |x - x_1|| \lesssim k(x, x_1)(1 + \log^+ |x|),$$

and the bounds in Lemma 4.2 and Corollary 5.6 we estimate (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi_1(\lambda) \lambda^{-\frac{1}{2}-} (\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}+} k(x, x_1) k(y, y_1),$$

$$|\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi_1(\lambda) k(x, x_1) k(y, y_1) (\langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha+} \lambda^{-\frac{1}{2}-\alpha-} (b - \lambda)^\alpha.$$

Applying Lemma 5.9 together with these bounds as in the proof of the previous lemma, we bound the λ -integral by

$$k(x, x_1) k(y, y_1) (\langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha+} \frac{1}{t^{1+\alpha}}.$$

Therefore,

$$(57) \lesssim t^{-1-\alpha} \int_{\mathbb{R}^4} k(x, x_1) k(y, y_1) (\langle y \rangle \langle y_1 \rangle \langle x \rangle \langle x_1 \rangle)^{\frac{1}{2}+\alpha+} v(x_1) |QD_0Q(x_1, y_1)| v(y_1) dx_1 dy_1 \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}},$$

since $\|v(x_1) k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha+}\|_{L^2_{x_1}} \lesssim 1$. \square

We now turn to the contribution of the error term $E^\pm(\lambda)$ from Lemma 4.6 in (37). Dropping the ‘ \pm ’ signs, we need to consider

$$(61) \quad \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda} \mathcal{E}(\lambda) v(x_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$\mathcal{E}(\lambda) := \chi_1(\lambda) R_0(\lambda^2)(x, x_1) E(\lambda)(x_1, y_1) R_0(\lambda^2)(y, y_1).$$

Proposition 5.15. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, then we have*

$$|(61)| \lesssim \frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}.$$

Proof. Let

$$T_0 := \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| + \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| + \sup_{0 < \lambda < b \lesssim \lambda < \lambda_1} \frac{\lambda^{\frac{1}{2}+\alpha}}{(b - \lambda)^\alpha} |\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(\lambda)|.$$

By Lemma 4.6, we see that T_0 is Hilbert-Schmidt on $L^2(\mathbb{R}^2)$, and hence we have the following bounds for the kernels

$$|E^\pm(\lambda)| \lesssim \lambda^{\frac{1}{2}} T_0, \quad |\partial_\lambda E^\pm(\lambda)| \lesssim \lambda^{-\frac{1}{2}} T_0,$$

$$|\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(\lambda)| \lesssim \lambda^{-\frac{1}{2}-\alpha}(b-\lambda)^\alpha T_0, \quad \text{if } 0 < \lambda < b \lesssim \lambda < \lambda_1.$$

Moreover, using Lemma 4.2 and Corollary 5.6, we have (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$\begin{aligned} |R_0(\lambda^2)(x, x_1)| &\lesssim (1 + |\log \lambda|)k(x, x_1)(1 + \log^+ |x|) \lesssim \lambda^{0-}k(x, x_1)\langle x \rangle^{0+}, \\ |\partial_\lambda R_0(\lambda^2)(x, x_1)| &\lesssim \lambda^{-1} + \lambda^{-\frac{1}{2}}\sqrt{\langle x \rangle \langle x_1 \rangle}, \\ |\partial_\lambda R_0(\lambda^2)(x, x_1) - \partial_\lambda R_0(b^2)(x, x_1)| &\lesssim (b-\lambda)^\alpha \left[\lambda^{-(1+\alpha)} + \lambda^{-\frac{1}{2}}|x-x_1|^{\frac{1}{2}+\alpha} \right]. \end{aligned}$$

Therefore we have the bounds (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$\begin{aligned} |\partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}-}(\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}}k(x, x_1)k(y, y_1)T_0(x_1, y_1), \\ |\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}-\alpha}(b-\lambda)^\alpha(\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}+\alpha}k(x, x_1)k(y, y_1)T_0(x_1, y_1). \end{aligned}$$

Applying Lemma 5.9 as in the proof of Proposition 5.14 above yields the claim of the proposition. \square

We close this section by noting that Proposition 5.10, Proposition 5.11, Proposition 5.14, and Proposition 5.15 yield Proposition 5.2 and hence Theorem 1.2.

6. ZERO NOT REGULAR

Finally we consider the evolution of (2) when zero is not a regular point of the spectrum of $H = -\Delta + V$. In particular, we prove Theorem 1.3. The discussion here is not completely self-contained, the development of the expansions as well as the spectral structure of $-\Delta + V$ at zero energy in two spatial dimensions are quite lengthy and technical. We direct the interested reader to [20, 13] for these details.

If zero is not regular, we cannot use the resolvent expansions employed in Sections 4 or 5. However, we can use the resolvent expansions developed in [13] for the Schrödinger evolution with obstructions at zero energy. The proof of these expansions need not be adjusted to prove the dispersive bounds for the wave equation, we cite them without proof. These expansions have roots in the work of Jensen and Nenciu, [20].

Formally, see Theorem 6.2 of [20], Section 5 of [13] or Definition 6.1 below, the different types of resonances are defined in terms of the non-invertibility of certain operators. Alternatively, one can characterize the obstructions in terms of certain spectral subspaces of $L^2(\mathbb{R}^2)$. The obstructions at zero energy can also be related to distributional solutions to $H\psi = 0$. If $\psi \in L^\infty(\mathbb{R}^2)$ but $\psi \notin L^p(\mathbb{R}^2)$ for any $p < \infty$ we say there is an s-wave resonance at zero. If $\psi \in L^p(\mathbb{R}^2)$ for all $p \in (2, \infty]$ we say there is a p-wave resonance at zero. Finally, if $\psi \in L^2(\mathbb{R}^2)$ there is an eigenvalue at zero.

We note that resonances at zero energy exist only for Schrödinger operators in dimensions $n \leq 4$ and can be characterized in terms of distributional solutions to $H\psi = 0$ where the appropriate space for ψ differs with the spatial dimension considered. For example, in dimensions three and four one has $\langle \cdot \rangle^{-\beta}\psi \in L^2(\mathbb{R}^n)$ where $\beta = \frac{1}{2}+$ in dimension three or $\beta = 0+$ in dimension four. In one spatial dimension, a resonance is characterized by the Wronskian of the Jost solutions, see [18].

We note that it was shown in [20, 13] that a distributional solution of $H\psi = 0$ in two spatial dimensions corresponding to a zero-energy resonance or eigenvalue must satisfy $\psi \in L^\infty(\mathbb{R}^2)$ and have the form

$$\psi(x) = c_0 + \frac{c_1 x_1 + c_2 x_2}{\langle x \rangle^2} + \Psi(x)$$

Here $\Psi \in L^2$, and we write $x = (x_1, x_2) \in \mathbb{R}^2$. The first term corresponds to a one-dimensional space of ‘s-wave’ resonance functions and the second term corresponds to a two-dimensional space of ‘p-wave’ resonances. One can see that if $c_0 \neq 0$ there is an s-wave resonance at zero, if $c_0 = 0$ but one of c_1 or c_2 is non-zero there is a p-wave resonance, and finally if $c_0 = c_1 = c_2 = 0$ there is an eigenvalue at zero.

We note that in the case of the free equation, when $V \equiv 0$, there is an s-wave resonance at zero energy due to the solution $\psi \equiv 1$ to $-\Delta\psi = 0$. In spite of this low energy obstruction, the free evolution still satisfies the $t^{-\frac{1}{2}}$ decay rate. As in the case of the Schrödinger equation, we see (in Theorem 1.3) that the s-wave resonance does not affect the natural rate of time decay.

The key difference from the previous sections when zero is not regular is that the expansions for $M^\pm(\lambda)^{-1}$ given in Lemmas 4.6 and 5.8 are no longer valid. The effect of such obstructions is a strictly low energy phenomenon, and the bounds attained in the high energy regime, Section 3, hold with no adjustments. To prove our main result, we need only adjust the low energy argument to account for resonances and/or eigenvalues at zero. The development of such expansions was first studied in the context of the Schrödinger operator e^{itH} in [20], adapted to the study of $L^1 \rightarrow L^\infty$ dispersive estimate when zero is regular in [31] and adapted to the case of zero not regular in [13]. We use these expansions in (5) to establish bounds for the wave equation.

The obstructive resonance and/or eigenvalues at zero cause the spectral measure to be more singular as $\lambda \rightarrow 0$. Roughly speaking, if there is an s-wave resonance, one has a most singular term of size $\log \lambda$ as $\lambda \rightarrow 0$, see Lemma 6.3 below. A p-wave resonance leads to singular terms of size $\lambda^{-2}(\log \lambda)^{-k}$, for $k = 1, 2, \dots$, while an eigenvalue leads to singular terms of the form λ^{-2} plus singular terms of the same form as when there is a p-wave resonance, see Lemma 6.7 below. As in the previous sections, we use the Stone formula and expansions for the resolvents to reduce the operator bounds to showing certain oscillatory integral bounds. In this section we work to bound the new terms, which are singular as $\lambda \rightarrow 0$, that arise in the expansion(s) for $M^\pm(\lambda)^{-1}$ and note that the remaining terms have clear analogues in the expansions of Section 4 and are bounded as in the cases handled previously.

To develop a more precise expansion for $M^\pm(\lambda)$. We define the operators G_j for $j = 1, 2$ (see (9))

$$(62) \quad G_1 f(x) = \int_{\mathbb{R}^2} |x - y|^2 f(y) dy,$$

$$(63) \quad G_2 f(x) = \frac{1}{8\pi} \int_{\mathbb{R}^2} |x - y|^2 \log |x - y| f(x) dy.$$

To make the above discussion more precise, we employ the terminology used in [13] for the Schrödinger evolution. Compare to Definition 4.5.

- Definition 6.1.** (1) *We say zero is a regular point of the spectrum of $H = -\Delta + V$ provided $QTQ = Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.*
- (2) *Assume that zero is not a regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of QTQ as an operator on $QL^2(\mathbb{R}^2)$. Then $QTQ + S_1$ is invertible on $QL^2(\mathbb{R}^2)$. Accordingly, we define $D_0 = (QTQ + S_1)^{-1}$ as an operator on $QL^2(\mathbb{R}^2)$. We say there is a resonance of the first kind at zero if the operator S_1TPTS_1 is invertible on $S_1L^2(\mathbb{R}^2)$.*
- (3) *We say there is a resonance of the second kind at zero if S_1TPTS_1 is not invertible on $S_1L^2(\mathbb{R}^2)$ but $S_2vG_1vS_2$ is invertible on $S_2L^2(\mathbb{R}^2)$, where S_2 is the Riesz projection onto the kernel of S_1TPTS_1 .*
- (4) *Finally, if $S_2vG_1vS_2$ is not invertible on $S_2L^2(\mathbb{R}^2)$, we say there is a resonance of the third kind at zero. We note that in this case the operator $S_3vG_2vS_3$ is always invertible on S_3L^2 , where S_3 is the Riesz projection onto the kernel of $S_2vG_1vS_2$ (see (6.41) in [20] or Section 5 of [13]).*

We note that a resonance of the first kind corresponds to an s-wave resonance only at zero. A resonance of the second kind corresponds to a p-wave resonance at zero but no eigenvalue at zero. There may or may not be an s-wave resonance at zero for a resonance of the second kind. Finally, a resonance of the third kind corresponds to an eigenvalue at zero. There may or may not be s-wave or p-wave resonances at zero in this case.

Note that we used the operator D_0 in Section 4 when zero was regular. This is not an abuse of notation since $S_1 = 0$ when there is no resonance at zero. We also note that QD_0Q is still absolutely bounded in these cases. We also define the operators

$$\begin{aligned} D_1 &:= (S_1TPTS_1 + S_2)^{-1} && \text{on } S_1L^2(\mathbb{R}^2), \\ D_2 &:= (S_2vG_1vS_2 + S_3)^{-1} && \text{on } S_2L^2(\mathbb{R}^2), \\ D_3 &:= (S_2vG_2vS_2)^{-1} && \text{on } S_3L^2(\mathbb{R}^2). \end{aligned}$$

The operators D_j and the projections S_j for $j = 1, 2, 3$ are all absolutely bounded operators. The invertibility of $S_1TPTS_1 + S_2$ and $S_2vG_1vS_2 + S_3$ is clear from the definition of the projections S_2 and S_3 . The invertibility of $S_2vG_2vS_2$ was shown in Lemma 5.4 in [13]. For a more complete discussion of these operators see Section 5 of [13]. In particular, these operators are absolutely bounded on $S_jL^2(\mathbb{R}^2)$ for $j = 1, 2, 3$ appropriately. Finally, we note that the projections $S_1 - S_2$, $S_2 - S_3$ and S_3 correspond to the s-wave resonances, p-wave resonances and zero eigenvalues respectively.

The characterization of resonances in Definition 6.1 is useful when trying to invert the operators $M^\pm(\lambda)$. To invert these operators one requires a technical inversion lemma, see Lemma 2.1 in [20] and a longer expansion for $M^\pm(\lambda)$ which we now state. By Lemma 2.2 of [13]

Lemma 6.2. *For $\lambda > 0$ define $M^\pm(\lambda) := U + vR_0^\pm(\lambda^2)v$. Let $P = v\langle \cdot, v \rangle \|V\|_1^{-1}$ denote the orthogonal projection onto v . Then*

$$M^\pm(\lambda) = g^\pm(\lambda)P + T + M_0^\pm(\lambda).$$

Here $g^\pm(\lambda) = a \ln \lambda + z$ where $a \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, and $T = U + vG_0v$ where G_0 is an integral operator defined in (33). Further, for any $\frac{1}{2} \leq k < 2$,

$$M_0^\pm(\lambda) = \tilde{O}_1(\lambda^k)$$

if $v(x) \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 1 + k$. Moreover,

$$(64) \quad M_0^\pm(\lambda) = g_1^\pm(\lambda)vG_1v + \lambda^2vG_2v + M_1^\pm(\lambda).$$

Here G_1, G_2 are integral operators defined in (62), (63), and $g_1^\pm(\lambda) = \lambda^2(\alpha \log \lambda + \beta_\pm)$ where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta_\pm \in \mathbb{C} \setminus \mathbb{R}$. Further, for any $2 < \ell < 4$,

$$M_1^\pm(\lambda) = \tilde{O}_1(\lambda^\ell)$$

if $\beta > 1 + \ell$.

The difference between this expansion and the previous expansion for $M^\pm(\lambda)$ in Lemma 4.3 is the more detailed expansion of the ‘error term’ $E_1^\pm(\lambda)$.

6.1. The case of an s-wave resonance. We first consider the case of a resonance of the first kind (an s-wave resonance at zero only), that is when there is a distributional solution to $H\psi = 0$ with $\psi \in L^\infty$ and $\psi \notin L^p$ for any $p < \infty$. As this is the type of resonance that occurs for the free operator, we expect to attain the $t^{-\frac{1}{2}}$ dispersive decay rate.

We note Corollary 2.7 in [13],

Lemma 6.3. *Assume that $|v(x)| \lesssim \langle x \rangle^{-1-k}$ for some $k \in [\frac{1}{2}, 2)$. Then in the case of a resonance of the first kind, we have*

$$\begin{aligned} M^\pm(\lambda)^{-1} &= -h_\pm(\lambda)S_1D_1S_1 - SS_1D_1S_1 - S_1D_1S_1S \\ &\quad - h_\pm(\lambda)^{-1}SS_1D_1S_1S + h_\pm(\lambda)^{-1}S + QD_0Q + \tilde{O}_1(\lambda^k), \end{aligned}$$

provided that λ is sufficiently small.

We can now express the perturbed resolvent as in (37) with the above expansion for $M^\pm(\lambda)^{-1}$ in place of the expansion used when zero is regular. This new expansion can be used in the Stone formula to deduce the desired bound in Theorem 1.3.

Roughly speaking, the analysis of the low energy contribution when zero is regular controls all by the the most singular terms in the above expression. That is, one needs to account for the $-h_{\pm}(\lambda)$ terms. Recall that $h_{\pm}(\lambda) = a \log \lambda + z$ so a mild singularity in the spectral measure is introduced as $\lambda \rightarrow 0$. To control the remaining less singular terms requires only very minor adjustments to Lemma 4.7, Proposition 4.8, Lemma 4.9 and Lemma 4.10. Accordingly we first control the new singular terms featuring $h_{\pm}(\lambda)$.

Proposition 6.4. *If $|V(x)| \lesssim \langle x \rangle^{-4-}$, we have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^{\infty} (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) \mathcal{K}(\lambda, |x - x_1|, |y - y_1|) \right. \\ \left. v(x_1) S_1 D_1 S_1(x_1, y_1) v(y_1) d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}},$$

where

$$(65) \quad \mathcal{K}(\lambda, p, q) = h^+(\lambda) H_0^+(\lambda p) H_0^+(\lambda q) - h^-(\lambda) H_0^-(\lambda p) H_0^-(\lambda q) \\ = 2ia \log(\lambda) [Y_0(\lambda p) J_0(\lambda q) + J_0(\lambda p) Y_0(\lambda q)] + 2z [J_0(\lambda p) J_0(\lambda q) + Y_0(\lambda p) Y_0(\lambda q)].$$

Proof. This proof follows along the lines of the proofs presented in Section 4 with modifications to account for the singularities in the spectral measure. The dispersive bound for the ‘low-low’ interaction follows from the arguments of Proposition 3.2 in [13] and the discussion in the proof of Proposition 4.8. Where stationary phase is used for Schrödinger, the wave equation admits the following argument, in a ‘high-low’ or ‘high-high’ interaction and occurs when $t - s \leq \frac{t}{2}$ where $s = p$, q or $p + q$, where $p = |x - x_1|$, $q = |y - y_1|$. Consider a ‘high-low’ interaction of the first term in (65), in the case that $t - p \leq \frac{t}{2}$ we have to bound

$$\left| \int_0^{\infty} (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) (\log \lambda) e^{-i\lambda p} \rho_+(\lambda p) F(\lambda, y_1, y) d\lambda \right| \\ \lesssim \int_0^{\lambda_1} \log(\lambda) \frac{1}{(1 + \lambda p)^{\frac{1}{2}}} d\lambda \lesssim \frac{1}{p^{\frac{1}{2}}} \int_0^{\lambda_1} \lambda^{-\frac{1}{2}-} d\lambda \lesssim p^{-\frac{1}{2}} \lesssim t^{-\frac{1}{2}}.$$

where we used that $p \gtrsim t$ in the last step. The ‘high-high’ interaction term can be similarly bounded when $t - (p + q) \geq \frac{t}{2}$. Without loss of generality, take $p \geq q$ then

$$\left| \int_0^{\infty} (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) (\log \lambda) e^{-i\lambda(p+q)} \rho_+(\lambda p) \rho_+(\lambda q) d\lambda \right| \\ \lesssim \int_0^{\lambda_1} \log(\lambda) \frac{1}{(1 + \lambda p)^{\frac{1}{2}}} d\lambda \lesssim \frac{1}{p^{\frac{1}{2}}} \int_0^{\lambda_1} \lambda^{-\frac{1}{2}-} d\lambda \lesssim p^{-\frac{1}{2}} \lesssim t^{-\frac{1}{2}}.$$

Here we used that $|\rho_+(\lambda q)| \lesssim 1$.

In the case when $t - p \geq \frac{t}{2}$ we integrate by parts. Due to the $\log \lambda$ term, we need to be slightly more careful and take advantage of the fact that $S_1 \leq Q$ we have $S_1 v = v S_1 = 0$. We recall the

following function and estimates from Lemma 3.7 of [13]. Define

$$(66) \quad \tilde{G}^\pm(\lambda, x, x_1) := \tilde{\chi}(\lambda|x-x_1|)\omega_\pm(\lambda|x-x_1|) - e^{\pm i\lambda(|x-x_1|-(1+|x|))}\tilde{\chi}(\lambda(1+|x|))\omega_\pm(\lambda(1+|x|)).$$

with ω_\pm as in (11). Then for any $0 \leq \tau \leq 1$ and $\lambda \leq 2\lambda_1$,

$$\begin{aligned} |\tilde{G}^\pm(\lambda, x, x_1)| &\lesssim (\lambda\langle x_1 \rangle)^\tau \left(\frac{\tilde{\chi}(\lambda|x-x_1|)}{(\lambda|x-x_1|)^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda(1+|x|))}{(\lambda(1+|x|))^{\frac{1}{2}}} \right), \\ |\partial_\lambda \tilde{G}^\pm(\lambda, x, x_1)| &\lesssim \langle x_1 \rangle \left(\frac{\tilde{\chi}(\lambda|x-x_1|)}{(\lambda|x-x_1|)^{\frac{1}{2}}} + \frac{\tilde{\chi}(\lambda(1+|x|))}{(\lambda(1+|x|))^{\frac{1}{2}}} \right) \end{aligned}$$

Taking $\tau > 0$, the smallness of \tilde{G}^\pm as $\lambda \rightarrow 0$ keeps the resulting function integrable if the derivative acts on the $\log \lambda$. According we bound with

$$\begin{aligned} \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} \left(\chi_1(\lambda) \log(\lambda) F(\lambda, x, x_1) \tilde{G}^\pm(\lambda, y, y_1) \right) \right| d\lambda &\lesssim \frac{k(x, x_1)\langle y_1 \rangle}{t} \int_0^{\lambda_1} 1 + |\log \lambda| + \lambda^{\tau-1} d\lambda \\ &\lesssim \frac{k(x, x_1)\langle y_1 \rangle}{t} \end{aligned}$$

Where we take $\tau = 0+$ in the bounds on \tilde{G} to ensure integrability. The full power of $\langle y_1 \rangle$ arises from when the derivative acts on \tilde{G}^\pm , or if one simply takes $\tau = 1$. A similar bound for the ‘high-high’ interaction when $t - (p+q) \geq \frac{t}{2}$ can be found by replacing F with \tilde{G} . It is this term that necessitates extra decay on the potential, as we now need $\langle y_1 \rangle v(y_1)$ to be in $L^2(\mathbb{R}^2)$, that is why we assume $v(y_1) \lesssim \langle y_1 \rangle^{-2-}$. □

Finally with $p = |x - x_1|$, $q = |x| + 1$ we define

$$G(\lambda, x, x_1) := \chi(\lambda p)J_0(\lambda p) - \chi(\lambda q)J_0(\lambda q).$$

Then for any $\tau \in [0, 1]$ and $\lambda \leq 2\lambda_1$ we have

$$(67) \quad |G(\lambda, x, x_1)| \lesssim \lambda^\tau \langle x_1 \rangle^\tau, \quad |\partial_\lambda G(\lambda, x, x_1)| \lesssim \langle x_1 \rangle^\tau \lambda^{\tau-1}.$$

These bounds were proven in Lemma 3.3 of [13] and allow us to extract further λ -smallness from the Bessel function as $\lambda \rightarrow 0$. This smallness is needed for the $SS_1D_1S_1$ and $S_1D_1S_1S$ terms that appear in the expansion of $M^\pm(\lambda)^{-1}$ given in Lemma 6.3.

Lemma 6.5. *We have the bound*

$$(68) \quad \sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty (\sin(t\lambda) + \lambda \cos(t\lambda)) \chi_1(\lambda) v(x_1) [SS_1D_1S_1(x_1, y_1) + S_1D_1S_1S(x_1, y_1)] v(y_1) \right. \\ \left. \left[R_0^+(\lambda^2)(x, x_1)R_0^+(\lambda^2)(y_1, y) - R_0^-(\lambda^2)(x, x_1)R_0^-(\lambda^2)(y_1, y) \right] d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}}.$$

Proof. This proof follows along the lines of the proof of Proposition 4.8. Due to the difference of the ‘+’ and ‘-’ resolvents we need only bound integrals of the form

$$(69) \quad \int_0^\infty \sin(t\lambda) \chi_1(\lambda) Y_0(\lambda|x-x_1|) v(x_1) S S_1 D_1 S_1(x_1, y_1) v(y_1) J_0(\lambda|y-y_1|) d\lambda,$$

along with a similar term with the Bessel functions J_0 and Y_0 switching roles. As usual, the sine term is the most delicate to control. The key difference from Proposition 4.8 is that we do not have a projection orthogonal to the span of v on both sides of the operator $S S_1 D_1 S_1$, the worst case is when $Y_0(\lambda|x-x_1|)$ is supported on $\lambda|x-x_1| \lesssim 1$ in the above integral. In this case we cannot replace Y_0 with $F(\lambda, x, x_1)$ to control the logarithmic singularity as $\lambda \rightarrow 0$. Instead we use (67) to gain integrability at zero.

As usual, we interpolate between two bounds to see the $t^{-\frac{1}{2}}$ decay rate. We bound Y_0 with (44) to see that

$$|(69)| \lesssim k(x, x_1) \int_0^{\lambda_1} (1 + |\log \lambda|) G(\lambda, y, y_1) d\lambda \lesssim k(x, x_1).$$

Here we took $\tau = 0$ in (67) since $\log \lambda$ is integrable at zero. On the other hand, we can integrate by parts against $\sin(t\lambda)$ to gain time decay.

$$\begin{aligned} |(69)| &\lesssim \frac{1}{t} \int_0^{\lambda_1} \left| \frac{d}{d\lambda} (Y_0(\lambda|x-x_1|) G(\lambda, y, y_1)) \right| d\lambda \\ &\lesssim \frac{k(x, x_1)}{t} \int_0^{\lambda_1} \lambda^{-1} |G(\lambda, y, y_1)| + (1 + |\log \lambda|) |\partial_\lambda G(\lambda, y, y_1)| d\lambda \lesssim \frac{k(x, x_1) \langle y_1 \rangle^\tau}{t}. \end{aligned}$$

Selecting $\tau > 0$ ensures that the boundary term vanishes as well as keeping the first term in the integrand integrable at $\lambda = 0$.

The remaining terms to consider, ‘low-high’ and ‘high-high’ terms follow from the proof in Proposition 4.8, we leave the details to the reader. □

We turn now to the terms in Lemma 6.3 with $h_\pm(\lambda)^{-1}$. These terms are controlled by either Lemma 4.9 or the following bound whose proof is identical.

Lemma 6.6. *We have the bound*

$$(70) \quad \left| \int_{\mathbb{R}^4} \int_0^\infty \sin(t\lambda) \chi_1(\lambda) \left(\frac{R_0^+(\lambda^2)(x, x_1) v(x_1) S S_1 D_1 S_1 S(x_1, y_1) v(y_1) R_0^+(\lambda^2)(y_1, y)}{h_+(\lambda)} - \frac{R_0^-(\lambda^2)(x, x_1) v(x_1) S S_1 D_1 S_1 S(x_1, y_1) v(y_1) R_0^-(\lambda^2)(y_1, y)}{h_-(\lambda)} \right) d\lambda dx_1 dy_1 \right| \lesssim t^{-\frac{1}{2}}$$

uniformly in x and y .

The analysis for the remaining terms in Lemma 6.3 are controlled as in Section 4, see Proposition 4.8, Lemmas 4.9 and 4.10. In particular one takes advantage of the identity $Qv = vQ = 0$ to

employ the functions $F(\lambda, x, x_1)$, $\tilde{G}(\lambda, x, x_1)$ and/or $G(\lambda, x, x_1)$ in the place of Y_0 and J_0 respectively. Finally we note that the statements of the propositions and lemmas hold if we replace $\sin(t\lambda)$ with $\cos(t\lambda)\lambda$ as these terms and their derivatives are even smaller. This suffices to prove the first statement in Theorem 1.3.

6.2. The case of a p-wave resonance and/or eigenvalue. We now consider the case when there are non-trivial solutions of $H\psi = 0$ with either $\psi \in L^p$ for all $p > 2$, that is a p-wave resonance, or $\psi \in L^p$ for all $p \geq 2$ an eigenvalue at zero. (Equivalently, when $S_1 T P T S_1$ is not invertible on $S_1 L^2$.) These types of resonances correspond to a loss of the time decay rate.

Let $\mathcal{D} = D_2 + S_3 D_3 S_3 v G_2 v S_2 D_2 S_2 v G_2 v S_3 D_3 S_3 - S_3 D_3 S_3 v G_2 v S_2 D_2 - D_2 S_2 v G_2 v S_3 D_3 S_3$. For the case of a p-wave resonance we note the expansion in Corollary 4.2 of [13],

Lemma 6.7. *Assume that $v(x) \lesssim \langle x \rangle^{-3-}$. Then, in the case of a resonance of the second kind, we have*

$$(71) \quad M^\pm(\lambda)^{-1} = \frac{S_2 D_2 S_2}{g_1^\pm(\lambda)} + Q\Gamma_1^\pm(\lambda)Q + Q\Gamma_2^\pm(\lambda) + \Gamma_3^\pm(\lambda)Q + \Gamma_4^\pm(\lambda) \\ + (M^\pm(\lambda) + S_1)^{-1} + \tilde{O}_1(\lambda^{2-}),$$

where Γ_i^\pm , $i = 1, 2, 3, 4$ are absolutely bounded operators on $L^2(\mathbb{R}^2)$ with $\Gamma_1^\pm(\lambda) = O(\lambda^{-2}(\log \lambda)^{-2})$, $\Gamma_2^\pm(\lambda), \Gamma_3^\pm(\lambda) = O(\lambda^{-2}(\log \lambda)^{-3})$, and $\Gamma_4^\pm(\lambda) = O(\lambda^{-2}(\log \lambda)^{-4})$.

In the case of a resonance of the third kind, we have

$$(72) \quad M^\pm(\lambda)^{-1} = \frac{S_3 D_3 S_3}{\lambda^2} + \frac{S_2 \mathcal{D} S_2}{g_1^\pm(\lambda)} + Q\Gamma_1^\pm(\lambda)Q + Q\Gamma_2^\pm(\lambda) + \Gamma_3^\pm(\lambda)Q + \Gamma_4^\pm(\lambda) \\ + (M^\pm(\lambda) + S_1)^{-1} + \tilde{O}_1(\lambda^{2-}),$$

where \mathcal{D} is as above, and Γ_i are absolutely bounded operators on $L^2(\mathbb{R}^2)$. These operators are distinct from the Γ_i in the case of a resonance of the second kind, but satisfy the same size estimates.

We note that the operators $(M^\pm(\lambda) + S_1)^{-1}$ have an expansion nearly identical in form to that we use in Lemma 4.6, see Lemma 2.5 in [13]. The error term is slightly different, but obeys the same error bounds. The resulting terms behave as in those already bounded in Section 4.

The new singular terms in (71) and (72) are all surrounded by projections which are orthogonal to the span of v , thus we can use the functions $F(\lambda, x, x_1)$, $G(\lambda, x, x_1)$, and $\tilde{G}(\lambda, x, x_1)$ defined in the previous section to gain extra λ smallness near $\lambda = 0$.

The solution operator can be shown to be bounded if there are p-wave resonances and/or eigenvalues at zero energy. The proof is nearly identical to that found in Section 4 of [13] for the Schrödinger equation. This follows as the oscillatory nature of the imaginary Gaussian $e^{it\lambda^2}$ in the integral is not used in [13] for these terms, and no integration by parts is used due to the highly singular nature of integrals as $\lambda \rightarrow 0$. In our case we do not use any oscillation of $\cos(t\lambda)$. We provide the details

for the convenience of the reader. We first establish the bounds for the cosine operator, then describe how one can reduce the contribution of the sine operator to the bounds for the cosine operator.

Accordingly, it suffices to establish the estimates for the contributions of the terms:

$$\frac{S_2 D_2 S_2}{g_1^\pm(\lambda)} + Q\Gamma_1(\lambda)Q + Q\Gamma_2(\lambda) + \Gamma_3(\lambda)Q + \Gamma_4(\lambda).$$

For the rest of the analysis, the standing assumption is that the potential satisfies the decay $|V(x)| \lesssim \langle x \rangle^{-6-}$. We start with the following.

Lemma 6.8. *We have the bound*

$$(73) \quad \sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) \left[\frac{R_0^+(\lambda^2)(x, x_1)v(x_1)S_2 D_2 S_2(x_1, y_1)v(y_1)R_0^+(\lambda^2)}{g_1^+(\lambda)} \right. \right. \\ \left. \left. - \frac{R_0^-(\lambda^2)(x, x_1)v(x_1)S_2 D_2 S_2(x_1, y_1)v(y_1)R_0^-(\lambda^2)(y_1, y)}{g_1^-(\lambda)} \right] d\lambda dx_1 dy_1 \right| \lesssim 1.$$

Proof. We note that we must exploit some cancellation between the ‘+’ and ‘-’ terms. Recall that $H_0^\pm(y) = J_0(y) \pm iY_0(y)$ and the definition of $g_1^\pm(\lambda)$ in Lemma 4.3 give us

$$(74) \quad \frac{R_0^+(\lambda^2)R_0^+(\lambda^2)}{g_1^+(\lambda)} - \frac{R_0^-(\lambda^2)R_0^-(\lambda^2)}{g_1^-(\lambda)} = \frac{J_0(\lambda p)J_0(\lambda q) - Y_0(\lambda p)Y_0(\lambda q)}{\lambda^2[(\log \lambda + c_1)^2 + c_2^2]} \\ + \frac{(J_0(\lambda p)Y_0(\lambda q) + Y_0(\lambda p)J_0(\lambda q))(\log \lambda + c_1)}{\lambda^2[(\log \lambda + c_1)^2 + c_2^2]}$$

We again must consider cases based on the supports of the resolvents. That is, we need to consider ‘low-low’, ‘high-low’ and ‘high-high’ interactions. Let us first consider the case when both resolvents are supported on the low part of their arguments. Contribution of the first term in (74) satisfies the required bound since $|J_0(z)| \lesssim 1$, and $\frac{1}{\lambda(\log \lambda)^2}$ is integrable on $[0, \lambda_1]$. Since the other terms have additional powers $\log \lambda$ in the numerator, we need to use the relation $S_2 v = v S_2 = 0$ (recall that $S_2 \leq Q$).

Consider the contribution of the second term in (74). We replace χY_0 with $F(\lambda, \cdot, \cdot)$, and using the bounds on F , we see

$$\left| \int_0^\infty \chi_1(\lambda) \frac{F(\lambda, x, x_1)F(\lambda, y, y_1)}{\lambda[(\log \lambda + c_1)^2 + c_2^2]} d\lambda \right| \lesssim k(x, x_1)k(y, y_1) \int_0^{\lambda_1} \frac{1}{\lambda(\log \lambda)^2} d\lambda \lesssim k(x, x_1)k(y, y_1).$$

The mixed J_0 and Y_0 terms in the second part of (74) are bounded similarly using $|G(\lambda, x, x_1)| \lesssim \lambda^{0+} \langle x_1 \rangle^{0+}$ to ensure integrability.

When one or both of the Bessel functions is supported on high energies, we use the functions $\tilde{G}(\lambda, p, q)$. The bound $|\tilde{G}(\lambda, x, x_1)| \lesssim \lambda^{0+} \langle x_1 \rangle^{0+}$ suffices for obtaining the required bound. The details are left to the reader. \square

Lemma 6.9. *For $\mathcal{C}_i(z) = J_0(z)$ or $Y_0(z)$ for $i = 1, 2$, we have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) \mathcal{C}_1(\lambda|x - x_1|)v(x_1) \right.$$

$$Q\Gamma_1(\lambda)Q(x_1, y_1)v(y_1)\mathcal{C}_2(\lambda|y - y_1|) d\lambda dx_1 dy_1 \Big| \lesssim 1.$$

Proof. Unlike in Lemma 6.8 we do not need to use any cancellation between the ‘+’ and ‘-’ terms. We consider the terms that arise when both \mathcal{C}_1 and \mathcal{C}_2 are supported on small energies. Consider,

$$\int_0^\infty \cos(t\lambda)\lambda\chi_1(\lambda)\chi(\lambda p)\mathcal{C}_1(\lambda p)vQ\Gamma_1(\lambda)Qv\chi(\lambda q)\mathcal{C}_2(\lambda q) d\lambda,$$

where $p = |x - x_1|$, $q = |y - y_1|$. In the worst case when $\mathcal{C}_1 = \mathcal{C}_2 = Y_0$, as $Qv = vQ = 0$, we replace χY_0 with F to obtain

$$(75) \quad \left| \int_0^{\lambda_1} \lambda F(\lambda, x, x_1)\Gamma_1(\lambda)F(\lambda, y, y_1) d\lambda \right| \\ \lesssim \left| \int_0^{\lambda_1} \frac{F(\lambda, x, x_1)F(\lambda, y, y_1)}{\lambda(\log \lambda)^2} d\lambda \right| \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2\Gamma_1(\lambda)| \\ \lesssim k(x, x_1)k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2\Gamma_1(\lambda)|.$$

The last line follows since $\sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2\Gamma_1(\lambda)|$ defines a bounded operator on $L^2(\mathbb{R}^2)$ (by Lemma 6.7), we are done. The other low energy terms are similar using G instead of F .

For the large energies, we note that the argument runs in a similar manner. Using $\tilde{\chi}(z)(|J_0(z)| + |Y_0(z)|) \lesssim 1$, and an argument as in (75), it easily follows that the integral is bounded as desired. \square

We need the following modified version of Lemma 6.9 to control the other $\Gamma_i(\lambda)$ terms, which don’t have the projection Q on both sides. It is worth noting that the loss of a Q on either side occurs exactly with the gain of a $(\log \lambda)^{-1}$ smallness as $\lambda \rightarrow 0$, which is exactly the gain one gets from replacing Y_0 with F .

Corollary 6.10. *For $\mathcal{C}_i(z) = J_0(z)$ or $Y_0(z)$ for $i = 1, 2$, we have the bound*

$$\sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty \cos(t\lambda)\lambda\chi_1(\lambda)\mathcal{C}_1(\lambda|x - x_1|)v(x_1)Q\Gamma_2(\lambda)(x_1, y_1)v(y_1)\mathcal{C}_2(\lambda|y - y_1|) d\lambda dx_1 dy_1 \right| \lesssim 1.$$

The same bounds hold when $Q\Gamma_2(\lambda)$ is replaced by $\Gamma_3(\lambda)Q$ or $\Gamma_4(\lambda)$.

Proof. We repeat the analysis of Lemma 6.9. Consider the case when both $\mathcal{C}_i(\lambda \cdot)$ are supported on low energies and both are Y_0 . We note that when $\lambda < 1$, we have

$$(76) \quad |Y_0(\lambda p)\chi(\lambda p)| \lesssim (1 + |\log \lambda|)(1 + \log^- p).$$

Using this and replacing χY_0 with F on one side, we obtain the bound

$$\int_0^{\lambda_1} \frac{|F(\lambda, x, x_1)|(1 + \log^- q)}{\lambda|\log \lambda|^2} d\lambda \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^3\Gamma_2(\lambda)| \\ \lesssim k(x, x_1)k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^3\Gamma_2(\lambda)|.$$

The same bound holds for $\Gamma_3(\lambda)Q$. For the contribution of $\Gamma_4(\lambda)$, we have

$$\begin{aligned} & \left| \int_0^\infty \lambda \chi(\lambda) Y_0(\lambda p) \Gamma_4(\lambda) Y_0(\lambda q) d\lambda \right| \\ & \lesssim \int_0^{\lambda_1} \frac{(1 + |\log \lambda|)(1 + \log^- p)(1 + |\log \lambda|)(1 + \log^- q)}{\lambda |\log \lambda|^4} d\lambda \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^4 \Gamma_4(\lambda)| \\ & \lesssim k(x, x_1) k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^4 \Gamma_4(\lambda)|. \end{aligned}$$

The other cases are similar.

When one of the $\mathcal{C}_i(\lambda \cdot)$ is supported on high energies, the analysis is less delicate. The required bound follows from $\tilde{\chi}(z)(|J_0(z)| + |Y_0(z)|) \lesssim 1$. \square

This completes the proof in the case of a resonance of the second kind.

We note that the above bounds in Lemma 6.9 and Corollary 6.10 also hold for the Γ_i term in (72). Thus for a resonance of the third kind, it suffices to consider the leading λ^{-2} term in (72). Noting (65) and the fact that the kernel of D_3 is real-valued, the following lemma completes the analysis.

Lemma 6.11. *We have the bound*

$$(77) \quad \sup_{x, y \in \mathbb{R}^2} \left| \int_{\mathbb{R}^4} \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) J_0(\lambda|x - x_1|) v(x_1) \frac{S_3 D_3 S_3}{\lambda^2} v(y_1) Y_0(\lambda|y - y_1|) d\lambda dx_1 dy_1 \right| \lesssim 1.$$

Proof. We provide a sketch of the proof. Due to similarities to previous proofs, we leave some details to the reader. We again consider the case when the Bessel functions are supported on low energy first. Accordingly, we wish to control

$$\begin{aligned} & \left| \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) \chi(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \chi(\lambda q) Y_0(\lambda q) d\lambda \right| \\ & \lesssim \left| \int_0^\infty \chi_1(\lambda) \frac{G(\lambda, x, x_1) F(\lambda, y, y_1)}{\lambda} d\lambda \right| \lesssim \langle x_1 \rangle^\tau k(y, y_1). \end{aligned}$$

Where we used $S_3 \leq Q$ and the known bounds for G with $\tau > 0$.

For the case when one function is supported on high energy, we have

$$\begin{aligned} & \left| \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) \tilde{\chi}(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \chi(\lambda q) Y_0(\lambda q) d\lambda \right| \\ & \lesssim \left| \int_0^\infty \chi_1(\lambda) \frac{\tilde{G}(\lambda, x, x_1) F(\lambda, y, y_1)}{\lambda} d\lambda \right| \lesssim \langle x_1 \rangle^\tau k(y, y_1). \end{aligned}$$

Similarly one uses $\tilde{G}(\lambda, y, y_1)$ instead of $F(\lambda, y, y_1)$ if we have $\tilde{\chi}(\lambda q)$.

When both functions are supported on high energy, we have

$$\begin{aligned} & \left| \int_0^\infty \cos(t\lambda) \lambda \chi_1(\lambda) \tilde{\chi}(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \tilde{\chi}(\lambda q) Y_0(\lambda q) d\lambda \right| \\ & \lesssim \left| \int_0^\infty \chi(\lambda) \frac{\tilde{G}(\lambda, x, x_1) \tilde{G}(\lambda, y, y_1)}{\lambda} d\lambda \right| \lesssim \langle x_1 \rangle^\tau \langle y_1 \rangle^\tau. \end{aligned}$$

\square

We are now ready to prove the main theorem of this section.

Theorem 6.12. *Let $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ be such that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 6$. Further assume that $H = -\Delta + V$ has a resonance of the second or third kind at zero energy. Then, there is a time dependent operator F_t such that*

$$\sup_t \|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1, \quad \|\cos(t\sqrt{H})P_{ac} - F_t\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-1}, \quad |t| > 1.$$

Proof. If we denote the terms that arise from the contribution of the terms in the first lines of (71) and (72) as F_t , Lemmas 6.8, 6.9, and 6.11 and Corollary 6.10 show that

$$\sup_t \|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1.$$

As the remaining terms in (71) and (72) are identical in form to those that arise in the analysis of a resonance of the first kind, we can use the bounds from the previous subsection to establish the theorem. \square

The sine operator can be seen to obey similar bounds, but with a bound that, at worst, grows linearly in time. To see this, we need only bound the most singular terms of the sine evolution. We use the following inequality to reduce the analysis to terms of the form previously considered

$$\left| \int_0^\infty \sin(t\lambda) \frac{\mathcal{E}(\lambda)}{\lambda} d\lambda \right| \lesssim |t| \int_0^\infty |\mathcal{E}(\lambda)| d\lambda.$$

Where we used the crude bound $|\sin(t\lambda)| \leq |t\lambda|$.

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