Streamlines and Stream Functions

Consider the two-dimensional, steady-state flow of an incompressible fluid. The general continuity equation simplifies as follows:

$$\underbrace{\frac{\partial \overset{=0}{\not{\rho}}}{\partial t}}_{\substack{\text{Steady-state}\\ \text{conditions}}} = \underbrace{-\frac{\partial}{\partial x}\left(\not{\rho} V_x\right) - \frac{\partial}{\partial y}\left(\not{\rho} V_y\right)}_{\substack{\rho \text{ comes outside of partial derivative}\\ \text{and then cancels out of the equation because}\\ \text{fluid is incompressible}}} - \underbrace{\cancel{\frac{\partial}{\partial z}\left(\rho V_z\right)}}_{\substack{=0\\ \text{Two-dimensional}}} \quad \Rightarrow \quad \boxed{0 = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}} \qquad (1.1)$$

Thus, any velocity field that describes a physically possible two-dimensional, steady-state flow of an incompressible fluid must satisfy Eq. (1.1)

As an example, consider the following velocity field that supposedly describes the two-dimensional, steady-state flow of an incompressible fluid:

$$V_x = U_o x \quad \text{and} \quad V_y = V_o y$$

Is it physically possible, i.e. does this velocity field satisfy continuity? If not, can it modified so that continuity is satisfied?

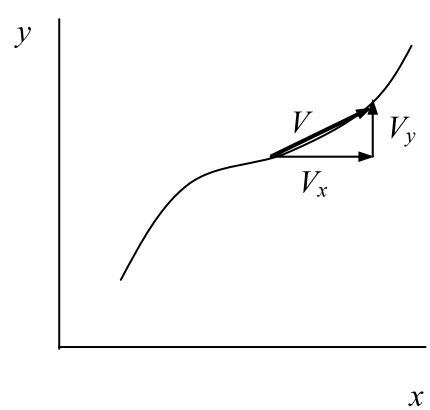
For the velocity field to satisfy continuity, it must satisfy the continuity equation simplified for the appropriate conditions. In this case, Eq. (1.1). To check this, we substitute the given velocity equations into Eq. (1.1) as follows:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial}{\partial x}\left(U_o x\right) + \frac{\partial}{\partial y}\left(V_o y\right) = U_o + V_o \neq 0$$

Since the equation does not equal zero, this velocity field *does not* satisfy continuity for a steady-state, two-dimensional flow of an incompressible fluid. However, it will satisfy continuity if $V_o = -U_o$. Thus, the following velocity field will satisfy continuity:

$$V_x = U_o x \ \text{ and } \ V_y = -U_o y$$

One way to visualize this flow is to sketch the velocity field as a forest of little arrows in an x-y grid each showing the direction and the magnitude of the velocity at a point in the flow field. However, for incompressible, steady-state, two-dimensional flow there is another tool that helps us visualize the flow. This tool is known as the **streamline** — a line in the flow field that is everywhere tangent to the velocity. Because a streamline is always tangent to the velocity at a every point along its length, there can be no flow *across* a streamline. The mathematical equation that describes the streamline in any flow is called a **stream function**.



To develop the equation for a streamline we must first describe the direction of the velocity at any point in the flow. At any point in the flow field, the direction of the velocity at any point can be described in terms of the x- and y-components of velocity. The slope of a line that is *tangent* to the velocity vector at a point (x, y)

can be written in terms of the finite displacement of a fluid particle at the point (*x,y*) over the time interval Δt:

$$\left.\begin{aligned}\Delta y &= \widetilde{V}_y \, \Delta t \\ \Delta x &= \widetilde{V}_x \, \Delta t\end{aligned}\right| \quad \rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\widetilde{V}_y \, \cancel{\Delta t}}{\widetilde{V}_x \, \cancel{\Delta t}} = \frac{\widetilde{V}_y}{\widetilde{V}_x} \tag{1.2}$$

where the terms with the tilde notation represent average velocities. In the limit as the time interval approaches zero, the *slope of a streamline* becomes

$$\frac{dy}{dx} = \frac{V_y}{V_x} \tag{1.3}$$

The stream function depends on both x and y: $\psi = \psi(x, y)$. For a function of two variables, the following is true:

$$\psi = \psi(x, y) \qquad \rightarrow \qquad d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{1.4}$$

However, the stream function has a constant value along any streamline. Thuse along any streamline, ψ is a constant and $d\psi = 0$:

$$d\psi = \left(\frac{\partial \psi}{\partial x}\right)_y dx + \left(\frac{\partial \psi}{\partial y}\right)_x dy = 0 \qquad \rightarrow \quad 0 = \left(\frac{\partial \psi}{\partial x}\right)_y dx + \left(\frac{\partial \psi}{\partial y}\right)_x dy \tag{1.5}$$

Comparing Eq. (1.5) for the stream function with the equation for the slope of a streamline, Eq. (1.3), gives the following

$$\begin{aligned} 0 &= \left(\frac{\partial \psi}{\partial x}\right)_y dx + \left(\frac{\partial \psi}{\partial y}\right)_x dy \\ \frac{dy}{dx} &= \frac{V_y}{V_x} \end{aligned} \quad \rightarrow \quad \begin{aligned} 0 &= \left(\frac{\partial \psi}{\partial x}\right)_y dx + \left(\frac{\partial \psi}{\partial y}\right)_x dy \\ V_x dy &= V_y dx \end{aligned}$$

$$\begin{aligned} 0 &= \left(\frac{\partial \psi}{\partial x}\right) dx + \left(\frac{\partial \psi}{\partial y}\right) dy \\ 0 &= \left(-V_y\right) dx + \left(V_x\right) dy \end{aligned} \tag{1.6}$$

Examining the two final equations above gives a relationship between the local velocities and the stream function:

$$\boxed{V_x = \left(\frac{\partial \psi}{\partial y}\right)_{\mathrm{x}} \qquad \text{and} \qquad V_y = -\left(\frac{\partial \psi}{\partial x}\right)_y} \tag{1.7}$$

This relation can be used in at least two different ways. Given a stream function, $\psi = \psi(x, y)$, the velocity at every point in the flow field can be calculated using Eq. (1.7). Alternatively, if we have a velocity field that describes a two-dimensional, steady-state, incompressible flow, then it should be possible to use Eq. (1.7) to determine the stream function that describes the flow.

Using the results from our earlier example, we can solve for the stream function that describes the flow as follows:

$$\left.\begin{array}{lll} V_x = U_o x = \left(\dfrac{\partial \psi}{\partial y}\right)_x & \rightarrow & \psi = U_o xy + f(x) \\ V_y = -U_o y = -\left(\dfrac{\partial \psi}{\partial x}\right)_y & \rightarrow & \psi = U_o xy + g(y) \end{array}\right| \rightarrow \begin{array}{c} \text{Only satisfied if} \\ f(x) = g(y) = C, \text{ a constant} \\ \psi = U_o xy + C \end{array}$$

Because the constant will be lost through the differentiation, there is no loss in generality if the constant $C = 0$. So the stream function for this flow is $\psi = U_o xy$. To sketch the streamlines, sketch y vs. x for a fixed value of ψ. Note that because there is no flow *across* a streamline, every streamline could be replaced by a solid boundary. Try sketching the streamline in the space below: