Conservation of Linear Momentum for a Differential Control Volume

When we applied the rate-form of the conservation of mass equation to a differential control volume (open system) in Cartesian coordinates, we obtained the **continuity equation**:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z}\right] \tag{1}$$

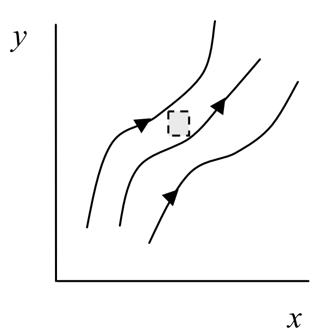
We now want to do the same thing for conservation of linear momentum. As with the development of the continuity equation, we will first apply the rate-form of conservation of linear momentum to a finite-size open system (a control volume) and then consider what happens in the limit as the size of the control volume approaches zero.

Conservation of Linear Momentum for a Two-Dimensional Flow

For simplicity, we will consider a two-dimensional, unsteady flow subject to a body force **b** as shown in Figure 1. The body force, the density, and the velocity all are field variables and may vary with position (*x, y*) and (*t*).

In words, the time rate of change of the linear momentum within the system equals the net transport rate of linear momentum into the system by external forces and by mass flow. To apply this principle, we select a small finite-size volume in the flow field, $\Delta x \Delta y \Delta z$, where Δz is the depth of the small volume into the paper, i.e. in the direction of the *z*-axis. Because of the multiple momentum transfers in a moving fluid, we will only focus on only one component of linear momentum at a time.

We will begin with the *x*-component of linear momentum, referred to hereafter as the "x-momentum." To identify the transfers of x-momentum, we enlarge the small volume and show *all* transfers of *x*-momentum on the diagram of the control volume (see Figure 2). Linear momentum can be transported into the system by surface forces acting on all four sides, by body forces within the system, and by mass flows across all four sides of the control volume.



Density field: $\rho = \rho(x, y, t)$

Velocity field: $\mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}}$

where $V_x = V_x(x, y, t)$ and $V_y = V_y(x, y, t)$

and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ = unit vectors for x, y axes

Body force field: $\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$

Figure 1 – A differential control volume for a two-dimensional flow

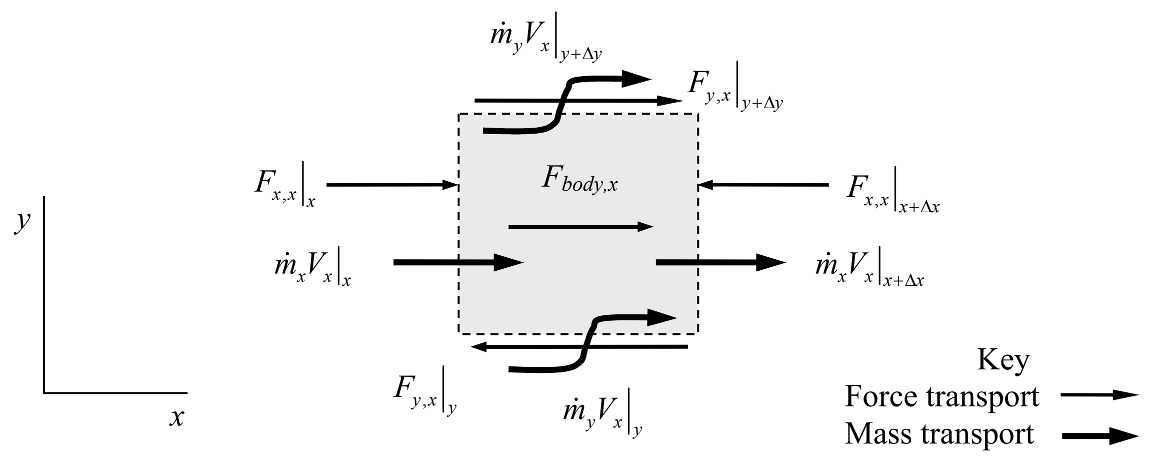


Figure 2 – Transport rates of the x-component of linear momentum for the differential control volume

Writing the rate-form of the conservation of linear momentum equation for the x-momentum gives:

$$\underbrace{\frac{dP_x}{dt}}_{\substack{\text{Rate of change}\\\text{of}\\\text{x-momentum}\\\text{inside the system}}} = \underbrace{F_{x,net}\Big|_{surface}}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{into the system}\\\text{by surface forces}}} + \underbrace{F_{x,net}\Big|_{body}}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{into the system}\\\text{by body forces}}} + \underbrace{\left(\dot{m}_x V_x\right)_{net}}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{into the system}\\\text{by mass flow}\\\text{in the x-direction}}} + \underbrace{\left(\dot{m}_y V_x\right)_{net}}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{into the system}\\\text{by mass flow}\\\text{in the y-direction}}} \qquad (2)$$

The left-hand side of Eq. (2) represents the rate of change (storage or accumulation). The terms on the right-hand side correspond to the transport rates of x-momentum illustrated in Figure 2.

<u>**Rate of change of x-momentum inside the control volume**</u>

The **x-momentum inside the control volume** can be written as follows using the mean-value theorem from calculus:

$$P_{x,sys} = \int_{\Delta x \Delta y \Delta z} V_x \rho \, d\forall = \widetilde{\rho V_x}\,\Delta x \Delta y \Delta z \qquad (3)$$

where the tilde notation $\widetilde{\rho V_x}$ indicates an average or mean value. The mean-value theorem says that the integral of a continuous function ρV_x over a volume equals the product of the volume and the mean value of the continuous function within the volume; thus, a simple product can replace the integral.

To find the **rate-of-change of the x-momentum inside the control volume**, the left-hand side of Eq. (2), we evaluate the derivative

$$\frac{dP_x}{dt} = \frac{d}{dt}\left(\widetilde{\rho V_x}\Delta x\Delta y\Delta z\right) = \left(\Delta x\Delta y\Delta z\right)\frac{\partial\left(\widetilde{\rho V}_x\right)}{\partial t} \tag{4}$$

Because we are holding *x, y,* and *z* constant during our differentiation the result is a partial derivative.

Net transport rate of x-momentum by external forces

The **net transport rate of x-momentum by body forces** is related to b_x , the component of the body force acting in the positive x-direction, and can be written as

$$\underbrace{F_{x,net}\Big|_{body}}_{\substack{\text{Net transport rate of x-momentum}\\ \text{into the system by body forces}}} = \int_{\Delta x\Delta y\Delta z} b_x \rho d\cancel{V} = \widetilde{\rho b_x}\left(\Delta x\Delta y\Delta z\right) \tag{5}$$

The gravitation field of the earth produces the most common body force; however, other body forces can also be important. For example, body forces produced by electrostatic fields can be used to enhance the flow of hot air over baked goods and speed cooking time. A body of fluid in a uniformly accelerating reference frame, for instance a glass of pop sitting on the floor of your car as it accelerates, can also be studied by treating the acceleration as a body force.

The **net transport rate of x-momentum by surface forces** is related to external forces acting on each surface of the control volume. For a two-dimensional flow, the surface force for any surface can be decomposed into two components, a normal force and a shear force. Each force, shear or normal, is defined as the integral of the appropriate surface stress over the surface area.

The **x-component of the surface force acting on the surface at** $x+\Delta x$, is wrtten as

$$F_{x,x}\Big|_{x+\Delta x} = \int_{\Delta y\Delta z} -\sigma_{xx} dA = -\left[\widetilde{\sigma_{xx}}\right]_{x+\Delta x}\Delta y\Delta z \tag{6}$$

where the minus sign occurs because the normal stress σ_{xx} by convention points out from the surface.

The **x-component of the surface force acting on the surface at** x , is written as

$$F_{x,x}\Big|_{x} = \int_{\Delta y\Delta z} -\sigma_{xx} dA = -\left[\widetilde{\sigma_{xx}}\right]_{x}\Delta y\Delta z \tag{7}$$

The **x-component of the surface force acting on the surface at *y*+Δ*y***, is written as

$$F_{y,x}\Big|_{y+\Delta y} = \int_{\Delta x\Delta z} \sigma_{yx} dA = \left[\widetilde{\sigma_{yx}}\right]_{y+\Delta y}\Delta x\Delta z \tag{8}$$

where the shear stress σ_{yx} points in the positive x-direction on a surface whose normal vector points in the positive y-direction.

The **x-component of the surface force acting on the surface at *y***, is written as

$$F_{y,x}\Big|_{y} = \int_{\Delta x\Delta z} \sigma_{yx} dA = \left[\widetilde{\sigma_{yx}}\right]_{y} \Delta x\Delta z \qquad (9)$$

where again the minus sign comes from the sign convention on the shear stress σ_{yx}.

Combining these forces to find the **x-component of the net surface force acting on the control volume** gives

$$\begin{aligned}
\underbrace{F_{x,net}\Big|_{surface}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by surface forces}}} &= \left\{ F_{x,x}\Big|_{x} - F_{x,x}\Big|_{x+\Delta x} \right\} + \left\{ F_{y,x}\Big|_{y+\Delta y} - F_{y,x}\Big|_{y} \right\} \\
&= \left\{\left[-\widetilde{\sigma_{xx}}\right]_{x} \Delta z\Delta y - \left[-\widetilde{\sigma_{xx}}\right]_{x+\Delta x} \Delta z\Delta y\right\} + \left\{\left[\widetilde{\sigma_{yx}}\right]_{y+\Delta y} \Delta z\Delta x - \left[\widetilde{\sigma_{yx}}\right]_{y} \Delta z\Delta x\right\} \\
&= \left[\widetilde{\sigma_{xx}}\Big|_{x+\Delta x} - \widetilde{\sigma_{xx}}\Big|_{x}\right]\left(\Delta z\Delta y\right) + \left[\widetilde{\sigma_{yx}}\Big|_{y+\Delta y} - \widetilde{\sigma_{yx}}\Big|_{y}\right]\left(\Delta z\Delta x\right) \qquad (10) \\
&= \left[\widetilde{\frac{\partial \sigma_{xx}}{\partial x}}\Delta x\right]\left(\Delta z\Delta y\right) + \left[\widetilde{\frac{\partial \sigma_{yx}}{\partial y}}\Delta y\right]\left(\Delta z\Delta x\right) \\
&= \left[\widetilde{\frac{\partial \sigma_{xx}}{\partial x}} + \widetilde{\frac{\partial \sigma_{yx}}{\partial y}}\right]\left(\Delta x\Delta y\Delta z\right)
\end{aligned}$$

<u>**Net transport rate of x-momentum by mass flow**</u>

Finally, we will examine the **net transport rate of x-momentum by mass flow**. Mass flow occurs on all four surfaces of the control volume. To begin, we will determine the x-momentum tranport rate of x-momentum into the system with mass flow at x and $x + \Delta x$:

$$\begin{aligned}
\underbrace{\left(\dot{m}_x V_x\right)_{net}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by mass flow}\\ \text{in the x-direction}}} &= \left(\dot{m}_x V_x\right)_{x} - \left(\dot{m}_x V_x\right)_{x+\Delta x} \\
&= \left[\overline{\left(\rho V_x \Delta z\Delta y\right)_{in} V_x}\right]_{x} - \left[\overline{\left(\rho V_x \Delta z\Delta y\right)_{out} V_x}\right]_{x+\Delta x} \qquad (11) \\
&= \left(\Delta z\Delta y\right)\left[\left(\widetilde{\rho V_x V_x}\right)_{x} - \left(\widetilde{\rho V_x V_x}\right)_{x+\Delta x}\right] \\
&= -\left(\Delta z\Delta y\right)\left[\frac{\partial \widetilde{\left(\rho V_x V_x\right)}}{\partial x}\Delta x\right]
\end{aligned}$$

Now we can determine the transport rate of x-momentum into the system with mass flow at y and $y + \Delta y$:

$$\underbrace{\left(\dot{m}_y V_x\right)_{net}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by mass flow}\\ \text{in the x-direction}}} = \left(\dot{m}_y V_x\right)_y - \left(\dot{m}_y V_x\right)_{y+\Delta y}$$

$$: \quad = \left[\overline{\left(\rho V_y \Delta z \Delta x\right)_{in} V_x}\right]_y - \left[\overline{\left(\rho V_y \Delta z \Delta x\right)_{out} V_x}\right]_{y+\Delta y} \tag{12}$$

$$= \left(\Delta z \Delta x\right)\left[\left(\widetilde{\rho V_y V_x}\right)_y - \left(\widetilde{\rho V_y V_x}\right)_{y+\Delta y}\right]$$

$$= -\left(\Delta z \Delta x\right)\left[\overline{\frac{\partial\left(\rho V_y V_x\right)}{\partial y}}\Delta y\right]$$

Putting it all together (the first time)

Now that we have developed expressions for each x-momentum transport or storage rate, we can substitute these values back into Eq. (2) repeated here for reference:

$$\underbrace{\frac{dP_x}{dt}}_{\substack{\text{Rate of change}\\ \text{of}\\ \text{x-momentum}\\ \text{inside the system}}} = \underbrace{F_{x,net}\Big|_{surface}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by surface forces}}} + \underbrace{F_{x,net}\Big|_{body}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by body forces}}} + \underbrace{\left(\dot{m}_x V_x\right)_{net}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by mass flow}\\ \text{in the x-direction}}} + \underbrace{\left(\dot{m}_y V_x\right)_{net}}_{\substack{\text{Net transport rate}\\ \text{of x-momentum}\\ \text{into the system}\\ \text{by mass flow}\\ \text{in the y-direction}}}$$

Making the substitutions for each term we have the following expression:

$$\underbrace{\left(\Delta x \Delta y \Delta z\right)\overline{\frac{\partial\left(\rho V_x\right)}{\partial t}}}_{\substack{\text{rate of change}\\ \text{of x-momentum}\\ \text{inside the system}}} = \underbrace{\left[\widetilde{\frac{\partial \sigma_{xx}}{\partial x}} + \widetilde{\frac{\partial \sigma_{yx}}{\partial y}}\right]\left(\Delta x \Delta y \Delta z\right)}_{\substack{\text{net transport rate of x-momentum}\\ \text{into the system by surface forces}\\ \text{(x-component of surface forces)}}} + \underbrace{\left(\widetilde{\rho b_x}\right)\left(\Delta x \Delta y \Delta z\right)}_{\substack{\text{net tranport rate of x-momentum}\\ \text{into the system by body forces}\\ \text{(x-component of body forces)}}} +$$
$$\underbrace{\left[-\overline{\frac{\partial\left(\rho V_x V_x\right)}{\partial x}}\left(\Delta x \Delta y \Delta z\right)\right]}_{\substack{\text{net transport rate of x-momentum}\\ \text{into the system}\\ \text{by mass flow in the x direction}}} + \underbrace{\left[-\overline{\frac{\partial\left(\rho V_y V_x\right)}{\partial y}}\left(\Delta x \Delta y \Delta z\right)\right]}_{\substack{\text{net transport rate of x-momentum}\\ \text{into the system}\\ \text{by mass flow in the y direction}}} \tag{13}$$

Dividing through by the volume $\Delta x \Delta y \Delta z$ and taking the limit as $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$, the average terms indicated by the tilde notation $\widetilde{(\ )}$ approach the value at the point (x,y,t).

This gives the **conservation of x-momentum for a differential control volume** (two-dimensional flow):

$$\frac{\partial(\rho V_x)}{\partial t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho b_x - \left[\frac{\partial(\rho V_x V_x)}{\partial x} + \frac{\partial(\rho V_y V_x)}{\partial y}\right] \tag{14}$$

A similar expression can be developed for the **conservation of y-momentum for a differential control volume** (two-dimensional flow):

$$\frac{\partial(\rho V_y)}{\partial t} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y - \left[\frac{\partial(\rho V_x V_y)}{\partial x} + \frac{\partial(\rho V_y V_y)}{\partial y}\right] \tag{15}$$

Revisiting the surface forces

Before we proceed, we need to re-examine the surface forces. Surface forces include both pressure forces or viscous forces. The pressure force is a normal force produced by the local pressure *P* acting over a surface. The local pressure is an intensive property of the substance and is a normal stress whose value is independent of orientation. Viscous forces are produced by viscous stresses τ_{ij} that depend on fluid viscosity and velocity gradients in the flow. Viscous stresses strongly depend on orientation and for any surface can be decomposed into both normal stresses and shear stresses.

For a two-dimensional flow the surface stress σ_{ij} can be separated into a pressure term *P* and a viscous stress term τ_{ij} as shown below:

$$\begin{aligned}\sigma_{ij} &= \begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \\ &= \begin{bmatrix} -P+\tau_{xx} & \tau_{yx} \\ \tau_{xy} & -P+\tau_{yy} \end{bmatrix} = \underbrace{\begin{bmatrix} -P & 0 \\ 0 & -P \end{bmatrix}}_{\text{Pressure Stress}} + \underbrace{\begin{bmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{bmatrix}}_{\text{Viscous Stress}}\end{aligned} \tag{16}$$

Now the surface stress terms can be rewritten to clearly separate the pressure and viscous stresses. For example in Eq. (14), the surface stress terms can be rewritten as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial}{\partial x}\left(-P+\tau_{xx}\right) + \frac{\partial \tau_{yx}}{\partial y} = \underbrace{-\frac{\partial P}{\partial x}}_{\text{Pressure}} + \underbrace{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}}_{\text{Viscous}} \tag{17}$$

Putting all together (again)

Substituting the appropriate value for each stress back into Equations (14) and (15) and rearranging terms gives the following expressions **conservation of linear momentum for a differential control volume in two-dimensional flow:**

x-component

$$\frac{\partial(\rho V_x)}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho b_x - \left[\frac{\partial(\rho V_x V_x)}{\partial x} + \frac{\partial(\rho V_y V_x)}{\partial y}\right]$$

(18)

y-component

$$\frac{\partial(\rho V_y)}{\partial t} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho b_y - \left[\frac{\partial(\rho V_x V_y)}{\partial x} + \frac{\partial(\rho V_y V_y)}{\partial y}\right]$$

Note that there is now a clear distinction between the pressure terms and the viscous stress terms. To evaluate the viscous stresses we need to have a constitutive model for the fluid that describes how the shear stresses are related to the velocity gradients in the flow and to the fluid property known as viscosity. When a flow is inviscid, it has no viscosity and the viscous stress terms disappear. This greatly simplifies the mathematics and can give useful results under some conditions.

Navier-Stokes Equations for Incompressible, Two-Dimensional Flow

Many important flows are essentially incompressible and this leads to significant simplifications. Additionally, we will restrict ourselves to Newtonian fluids. With these assumptions, we reproduce the Navier-Stokes equations for incompressible, two-dimensional flow.

To develop these equations, we first assume that the flow is incompressible and consider the consequences. The continuity equation reduces, Eq. (1) becomes

$$0 = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \tag{19}$$

In Eq. (18), the conservation of linear momentum equation, incompressible flow brings the density outside of the derivatives as shown below:

x-component

$$\rho \frac{\partial V_x}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho b_x - \rho \left[\frac{\partial (V_x V_x)}{\partial x} + \frac{\partial (V_y V_x)}{\partial y} \right]$$

y-component (20)

$$\rho \frac{\partial V_y}{\partial t} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho b_y - \rho \left[\frac{\partial (V_x V_y)}{\partial x} + \frac{\partial (V_y V_y)}{\partial y} \right]$$

Further simplifications occur by expanding the terms in brackets and applying the incompressible continuity equation. For the term in brackets in the x-component equation in Eq. (20), we have the following simplification

$$\begin{aligned} \frac{\partial (V_x V_x)}{\partial x} + \frac{\partial (V_y V_x)}{\partial y} &= V_x \frac{\partial V_x}{\partial x} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_x \frac{\partial V_y}{\partial y} \\ &= V_x \underbrace{\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right)}_{=0 \text{ Why?}} + \left(V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} \right) \\ &= \left(V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} \right) \end{aligned} \qquad (21)$$

Substituting this result and its counterpart for the y-component back into Eq. (20) gives **differential equation for conservation of linear in an incompressible, two-dimensional flow**:

x-component

$$\rho \frac{\partial V_x}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho b_x - \rho \left[V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} \right]$$

y-component (22)

$$\rho \frac{\partial V_y}{\partial t} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho b_y - \rho \left[V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} \right]$$

Please note the long list of qualifiers for these two equations. All of them are important and if used under *any* other conditions, Equation (22) will be in invalid.

A **Newtonian fluid** is a fluid in which viscous stresses are proportional to the rate of angular deformation within the fluid. The constant of proportionality is μ — the **dynamic viscosity**. The viscous stresses for an incompressible, two-dimensional flow of a Newtonian fluid become

$$\tau_{xx} = 2\mu \frac{\partial V_x}{\partial x}, \quad \tau_{yy} = 2\mu \frac{\partial V_y}{\partial x}, \quad \text{and} \quad \tau_{yx} = \tau_{xy} = \mu \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) \qquad (23)$$

Using these results, the viscous stress terms in Eq. (22) are replaced by terms containing the fluid viscosity and velocity gradients. Note viscous stresses depend on *both* the fluid and the flow.

The next step is to substitute the viscous stress terms in Eq. (23) back into Eq. (22) *and* use the incompressible continuity equation, Eq. (19), to simplify the expressions. Once done we recover the Navier-Stokes equations.

The **x-component of the Navier-Stokes Equation for a two-dimensional, incompressible flow** is shown below:

$$\underbrace{\rho\frac{\partial V_x}{\partial t}}_{\substack{\text{Rate of change}\\\text{of x-momentum}\\\text{per unit volume}}} = \underbrace{-\frac{\partial P}{\partial x}}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{by pressure forces}\\\text{per unit volume}}} + \underbrace{\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right)}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{by viscous forces}\\\text{per unit volume}}} + \underbrace{\rho b_x}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{by body forces}\\\text{per unit volume}}} - \underbrace{\rho\left(V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}\right)}_{\substack{\text{Net transport rate}\\\text{of x-momentum}\\\text{by mass flow}\\\text{per unit volume}}} \qquad (24)$$

This arrangement shows a clear connection to our original conservation of linear momentum equation; however, the more traditional arrangement found in most textbooks moves the mass transport term to the other side of the equal sign. When arranged in this form we have the result we have been seeking

Navier-Stokes Equations for a Two-Dimensional, Incompressible Flow

x-component

$$\rho\frac{\partial V_x}{\partial t} + \rho\left(V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \rho b_x + \mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right)$$

y-component

$$\rho\frac{\partial V_y}{\partial t} + \rho\left(V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}\right) = -\frac{\partial P}{\partial y} + \rho b_y + \mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right)$$

(25)

Following a similar process for three-dimensions, we can derive the full set of equations. These are stated below without explanation or development:

Navier-Stokes Equations for an Incompressible Flow

x-component: $\rho\frac{\partial V_x}{\partial t} + \rho\left(V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho b_x + \mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right)$

y-component: $\rho\frac{\partial V_y}{\partial t} + \rho\left(V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho b_y + \mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right)$

z-component: $\rho\frac{\partial V_z}{\partial t} + \rho\left(V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho b_z + \mu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2}\right)$